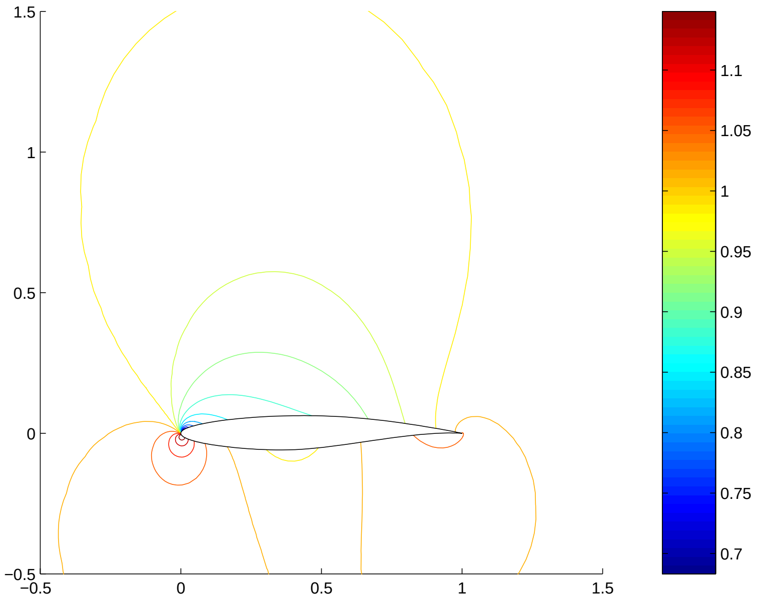
Investigation of Non-Linear Projection for POD Based Reduced Order Models for Aerodynamics



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Outline

- Motivation
- Proper Orthogonal Decomposition (POD) Theory
- Flow Analysis Procedure
- Results for Flow Analyses and Inverse Design Problem
- Conclusions and Future Work

Motivation

- There is a need for high-fidelity models in the multidisciplinary design and optimization of aerospace systems, but the computational cost is unfeasible for realistic problems.

- Response surfaces have disadvantages because the polynomial interpolate has no physical basis and there is no way to tell how well an approximate solution will agree with the exact solution.

- We are investigating POD as an alternate means to form approximate, reduced order models for use in the design environment.

- Initially we are investigating POD based models for use in Aerodynamic Shape Optimization problems but we are in the process of applying it to other problems where the method can be superior to response surfaces.

Proper Orthogonal Decomposition (POD)

- POD has its roots in statistical analysis and has appeared with various names, including: principal component analysis, empirical eigenfunctions, Karhunen-Loéve decomposition, and empirical orthogonal eigenfunctions.

- We are seeking representations of a function, $u(x)$, in terms of a basis $\{\varphi_j(x)\}_{j=1}^{\infty}$ which allows an approximation to be constructed as

$$u_M = \sum_{j=1}^{M} \eta_j \, \varphi_j(x) \tag{1}$$

- We would like to choose these basis functions such that they describe a typical function in the ensemble $\{\mathbf{u}^k\}$ better than any other linear basis.

POD Theory (cont.)

- These requirements may be shown to require that the basis functions satisfy

$$\int_{\Omega} \langle u(x)u(x')\rangle \varphi(x')dx' = \lambda\varphi(x). \tag{2}$$

- The POD basis is composed of the eigenfunctions of the integral Eq. 2.

- The member functions of the ensemble can now be decomposed as follows

$$u(x) = \sum_{j=1}^{\infty} \eta_j \; \varphi_j(x). \tag{3}$$

- The main advantage of the POD is that it produces the *best* linear representation for an ensemble of functions or flowfields (*snapshots*).

Flow Analysis Procedure

- Using the basis modes the flow solution can be expanded in the form

$$\mathbf{w}(\mathbf{x}) = \sum_{i=1}^{M} \eta_i \, \varphi_i(\mathbf{x}). \tag{4}$$

- Traditional uses of POD expansions in fluid dynamics have used this modal expansion to project the time evolution of the full incompressible Navier-Stokes equations.

- Our flow solutions are *steady* and, for airfoil analysis and design purposes, the design parameters, rather than time, will be identified with the coefficients of a surface parameterization that allows us to change the geometry.

Flow Analysis Procedure (cont.)

- We are seeking to expand solutions of the flow about arbitrary airfoil shapes using a linear superposition of the POD modes.

- We would like the resulting expansion to satisfy, as closely as possible, both the governing equations of the flow.

- The approach we have chosen to take is based on the well-known *finite-volume* procedure which is often used to discretize the governing equations of the flow to obtain a set of ordinary differential equations

$$\frac{d}{dt}(\mathbf{w}_{ij}\,V_{ij}) + \mathbf{R}(\mathbf{w}_{ij}) = \mathbf{0}. \qquad (5)$$

Flow Analysis Procedure (cont.)

- An exact solution using the reduced basis will typically not be possible since we have drastically reduced the number of available degrees of freedom.

- The expansion coefficients should therefore be computed in such a way that the governing equations are satisfied as closely as possible.

- We define the POD residual to be

$$\mathbf{R_{POD}}(\mathbf{w}_{ij}) = \mathbf{2} + \frac{\mathbf{R}^{+}(\mathbf{w}_{ij})}{\mathbf{R}^{-}(\mathbf{w}_{ij})} + \frac{\mathbf{R}^{-}(\mathbf{w}_{ij})}{\mathbf{R}^{+}(\mathbf{w}_{ij})} = \mathbf{0}, \tag{6}$$

where the $+$ and $-$ denote the positive and negative contributions to the residual.

Flow Analysis Procedure (cont.)

- Using the linear expansion, the flow variables in Eq. 6 can be considered functions of the expansion coefficients and the POD residual can be expressed as

$$\mathbf{R_{POD}}(\eta_l) = \mathbf{0}, \;\; l = 1, \ldots, M. \tag{7}$$

- An exact solution of Eq. 7 will usually not be possible so we define a POD cost function

$$I_{POD} = \sum_n \mathbf{R_{POD}}^2(\eta_l), \tag{8}$$

  to render the problem well posed and define the solution that for a given set of modes most closely satisfies the governing equations.

- The flow solution procedure consists of finding the least squares minimizer of Eq. 8, with solutions typically requiring 5 to 10 iterations.

Computational Costs

- The FLO82 flow solver, using a 5-stage Runge-Kutta scheme and multigrid, requires the equivalent of approximately *1,000* residual evaluations over the entire domain when artificial dissipation evaluations are also considered.

- The cost of the approximate flow solution from a set of basis modes is equivalent to about *50* evaluations of the residual over the entire domain.

- Over an order of magnitude decrease in computational cost is achieved for two-dimensional flows, and the reduction in three-dimensions is even greater.

Results - Flow Computations

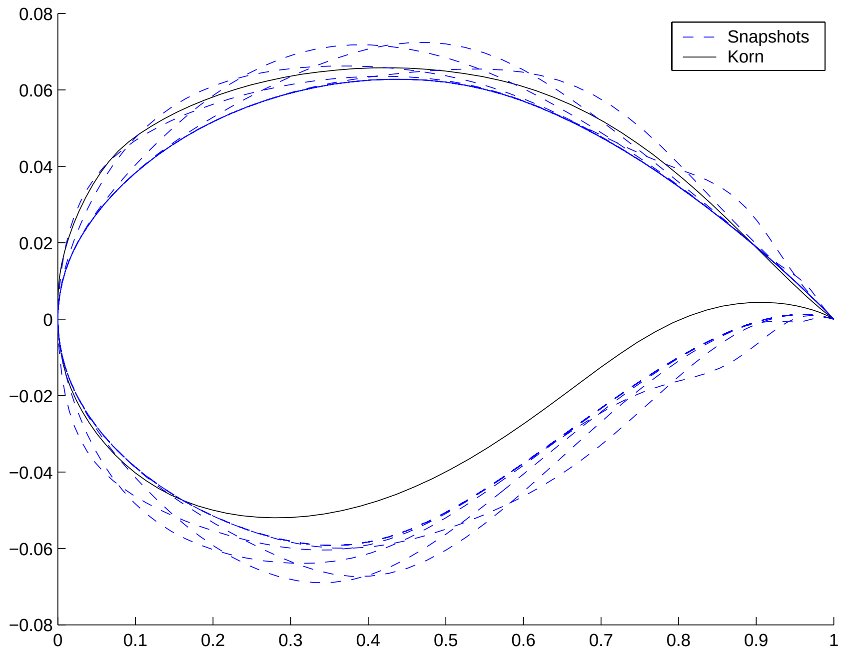
- We parameterize the RAE 2822 airfoil surface with a series of Hicks-Henne bump functions, which make smooth changes in the geometry.

- To define the snapshot geometries 14 bump functions, 7 each on the upper and lower surfaces, were used.

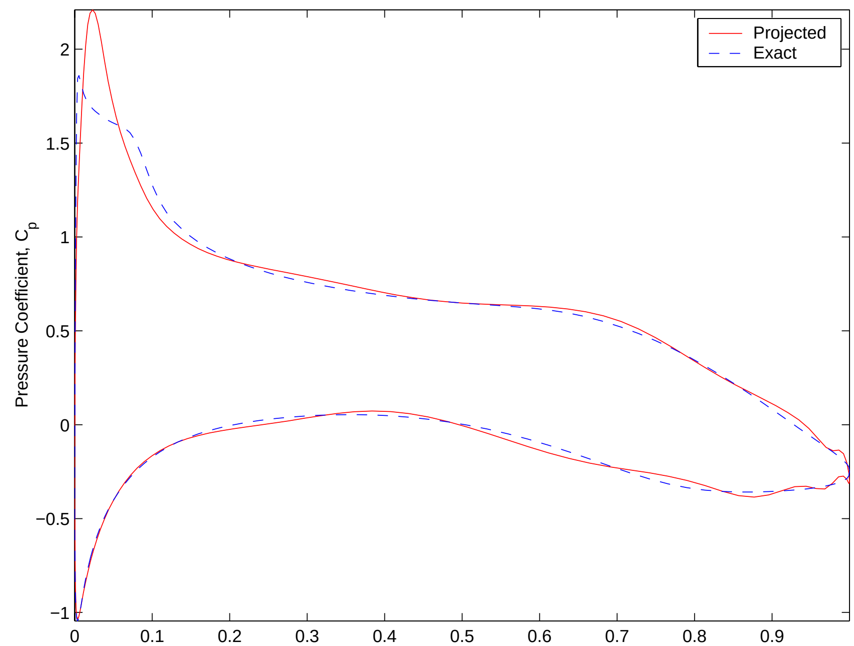
- Bump amplitude was 0.1% of the chord.

- Because any of the snapshots used in constructing a set of modes can be *exactly* represented by the modes, the projection algorithm should be able to compute the exact solution for any of the geometries represented in the snapshots.

Results - Projection of Korn, $M = 0.50$

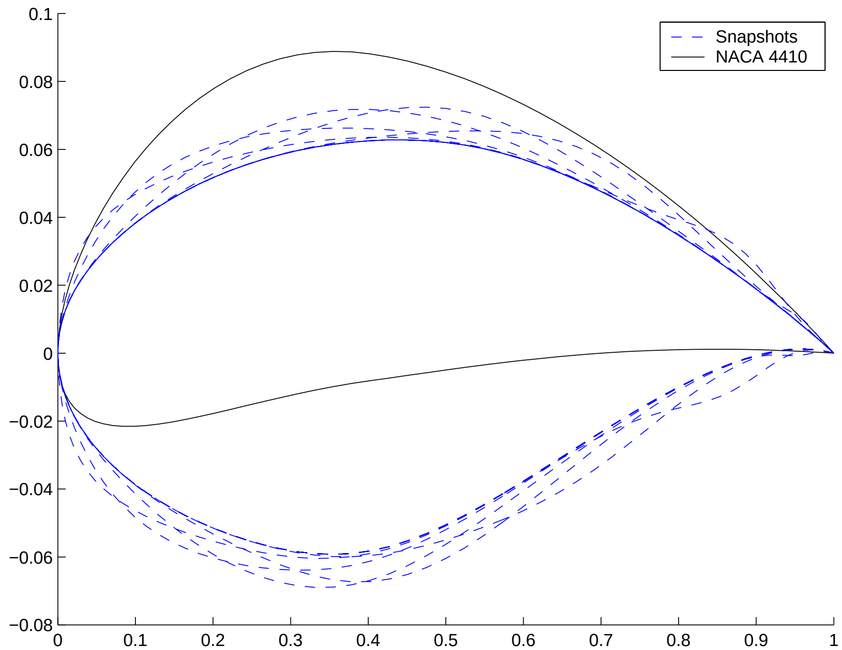


(a) Snapshots and Korn Airfoil

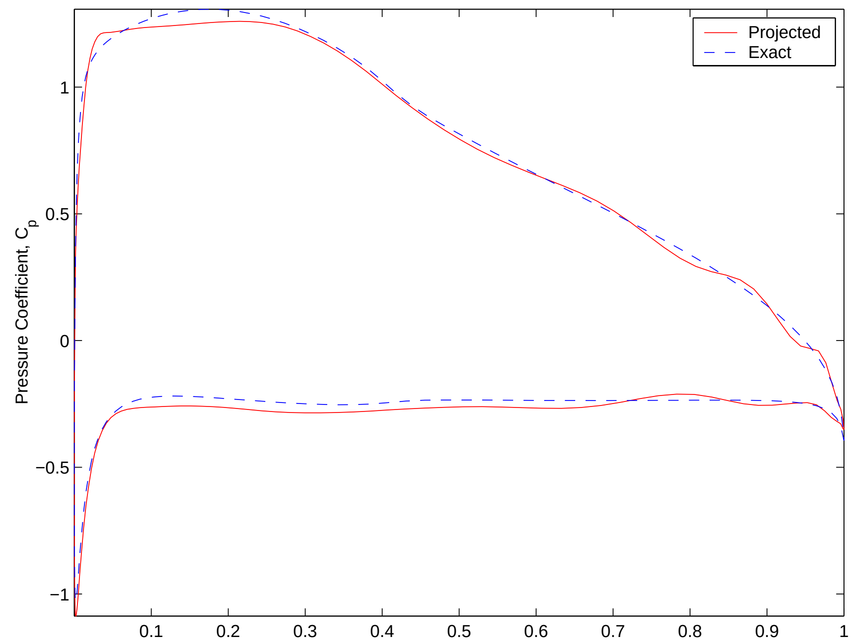


(b) C_P Comparison for Korn Airfoil

Results - Projection of NACA 4410, $M = 0.50$



(a) Snapshots and NACA 4410 Airfoil



(b) C_P Comparison for NACA 4410 Airfoil

Results - Inverse Design

- The use of POD models for design optimization was investigated by performing an inverse design. Although this problem can be efficiently treated using an adjoint formulation, we use it as a model problem.

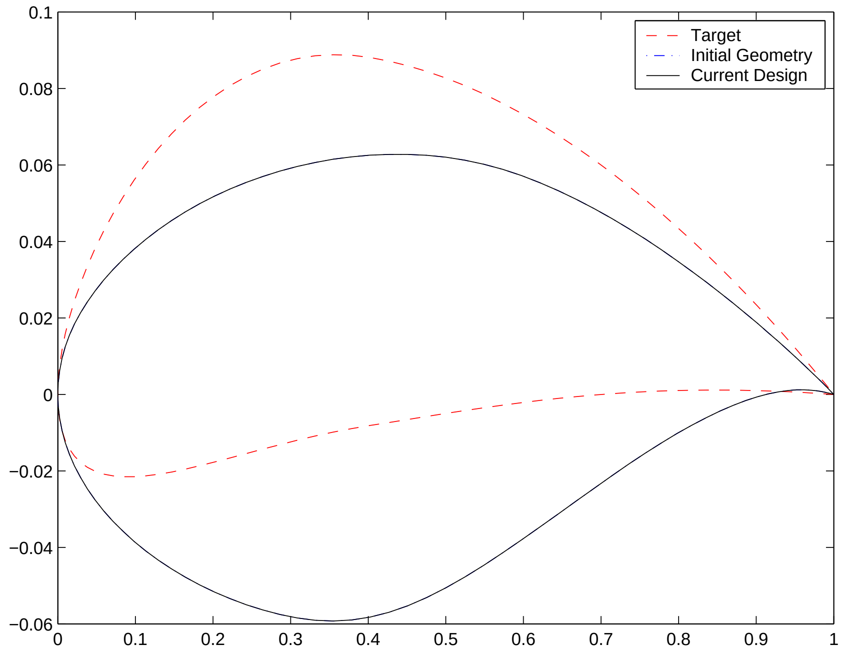
- An inverse design cost function is defined as

$$I_{ID} = \int_S (p - p_T)^2 ds. \tag{9}$$

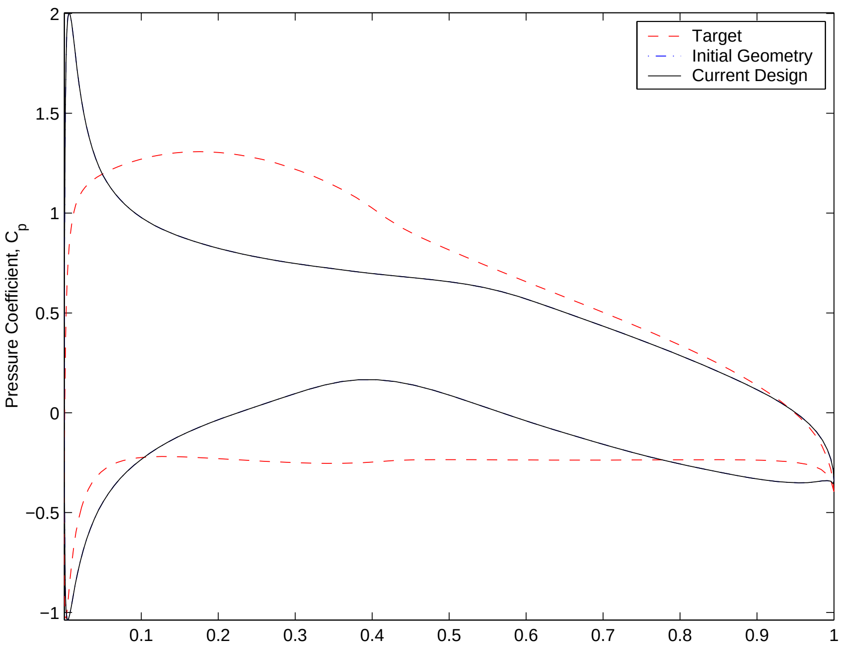
- Design variables were the amplitude of 20 bump functions.

- Gradients of the cost function were obtained by finite differencing of the reduced order model.

Results - Inverse Design (cont.)

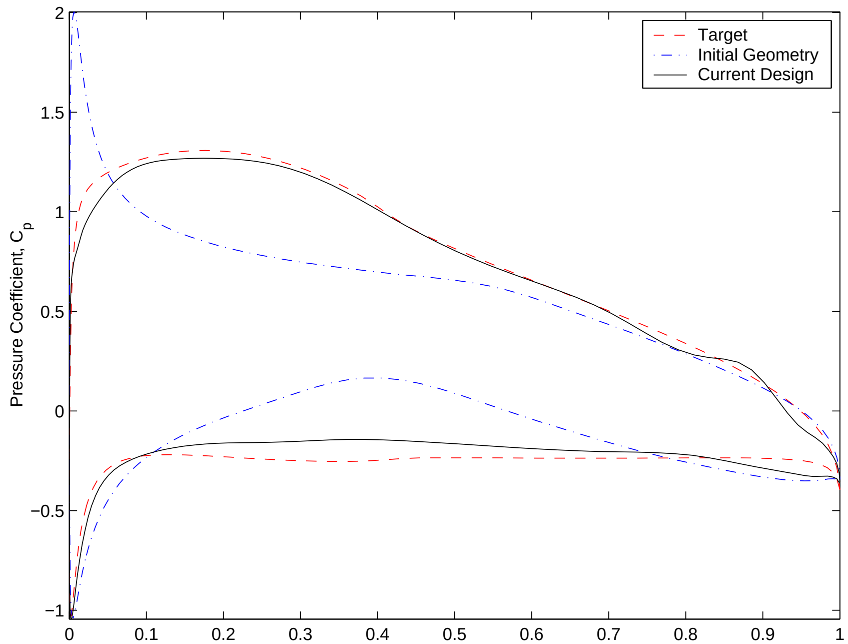


(a) Geometry for Initial Design

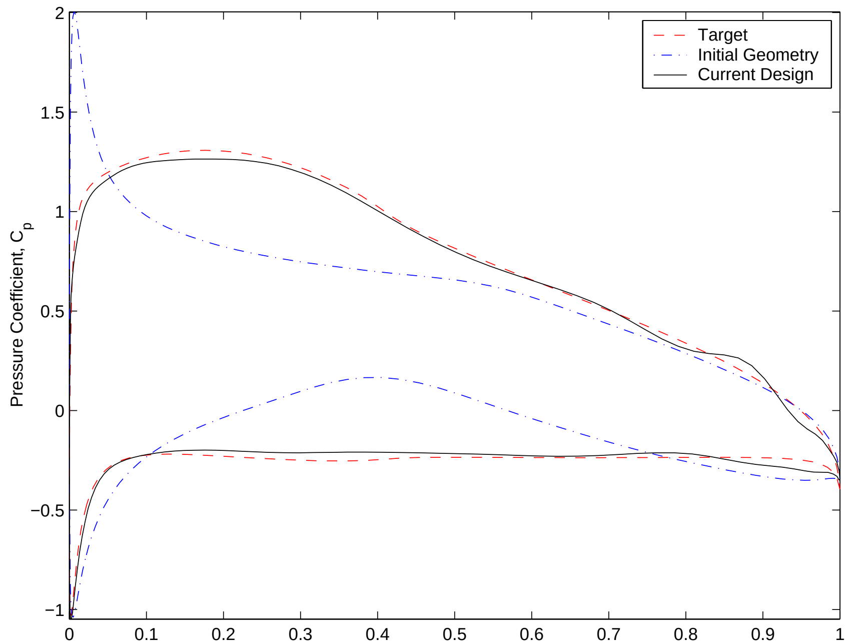


(b) Pressure Distribution for Initial Design

Results - Inverse Design (cont.)

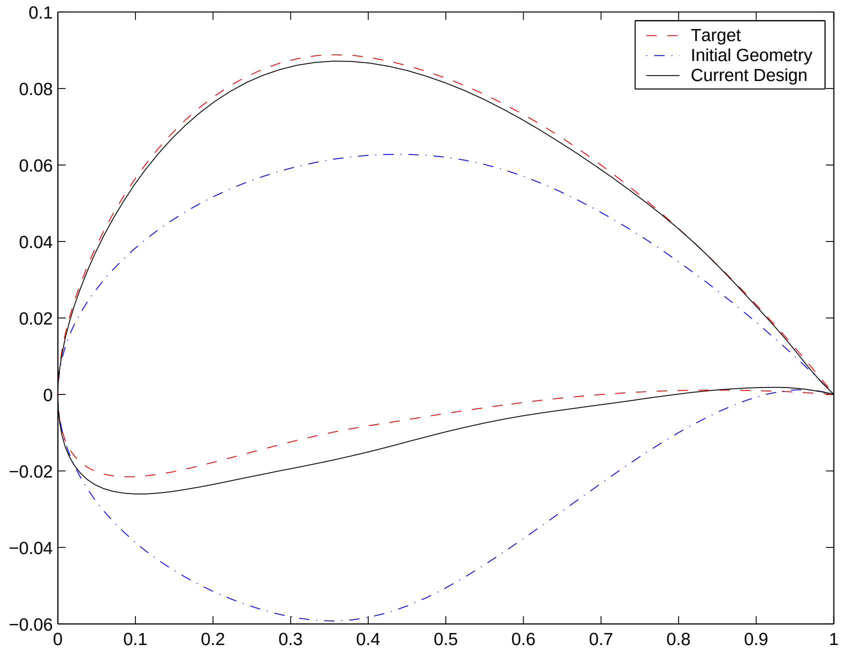


(a) Pressure at Design Iteration 5

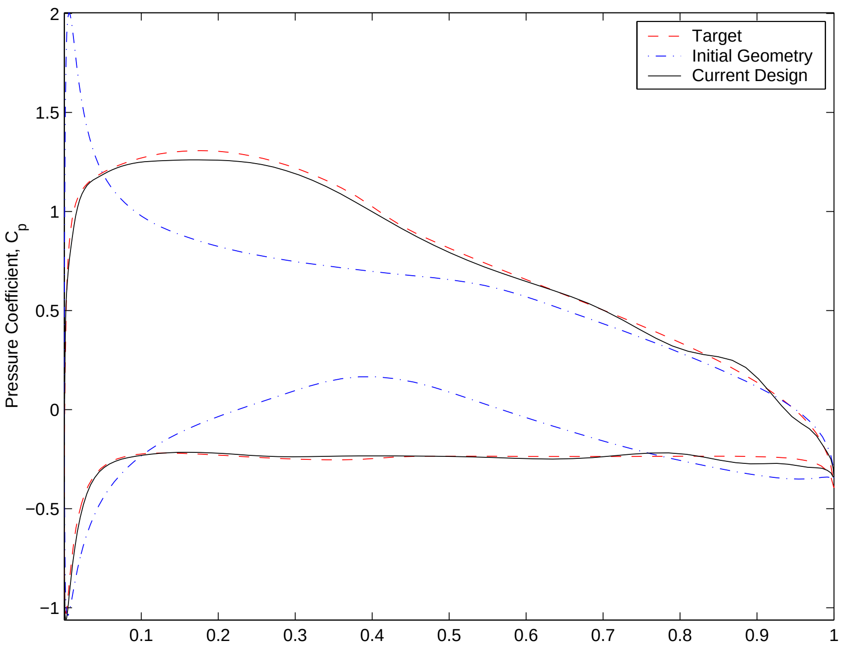


(b) Pressure at Design Iteration 10

Results - Inverse Design (cont.)

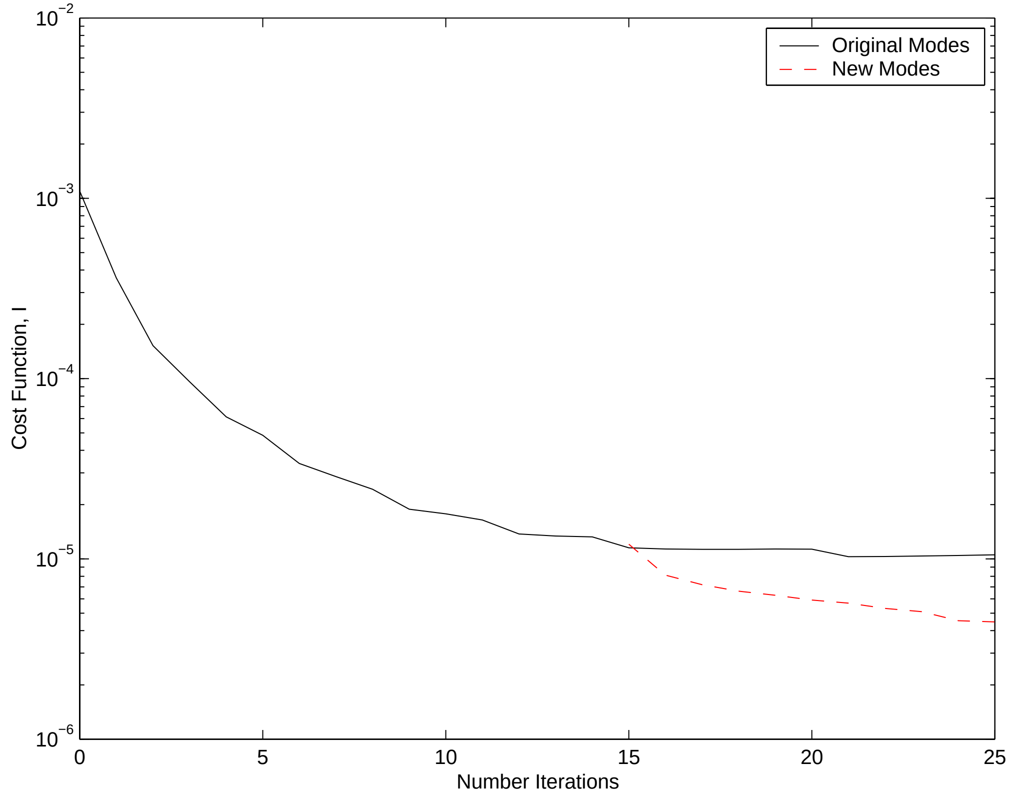


(a) Geometry at Design Iteration 15



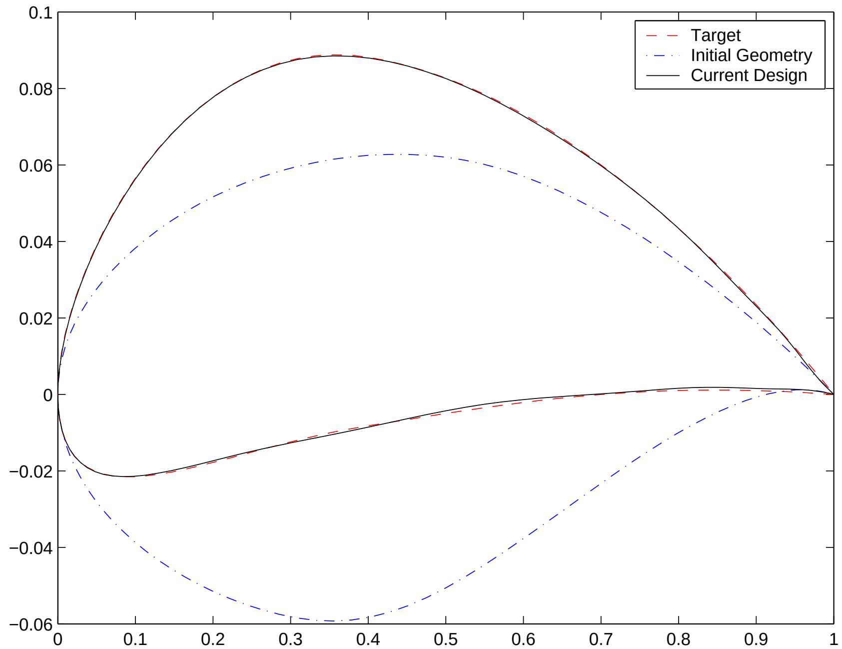
(b) Pressure at Design Iteration 15

Results - Inverse Design (cont.)

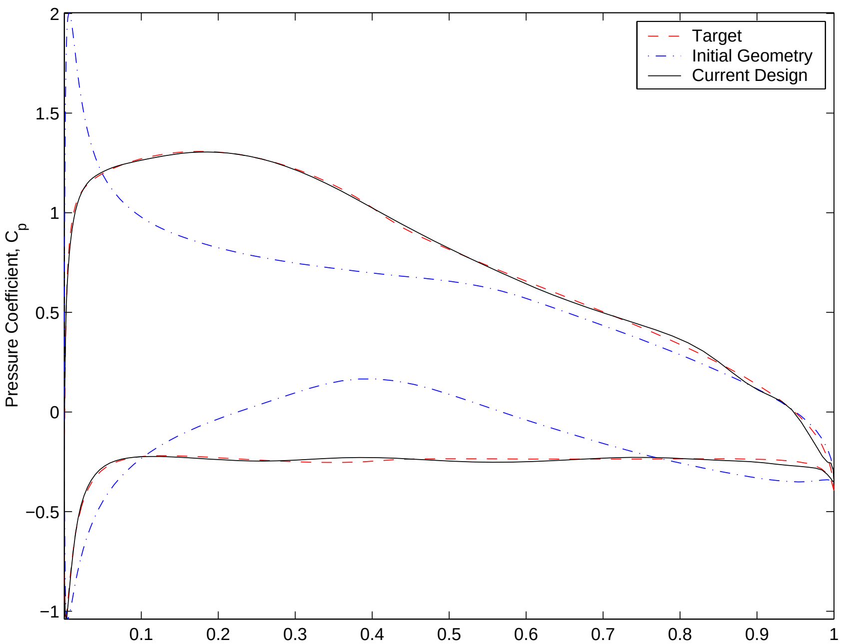


(a) Cost Function Convergence

Results - Inverse Design (cont.)



(a) Geometry at Design Iteration 25



(b) Pressure at Design Iteration 25

Conclusions and Future Work

- Computation of a flow solution is reduced to the solution of a small number of coupled non-linear equations.
- The computational cost is reduced by approximately an order of magnitude for 2-D flows, with potentially greater savings in 3-D.
- The design results show that a significant computational cost reduction is achieved with minimal degradation in accuracy.
- We are now applying this approach to non-linear transonic flows.
- Looking for design problems for which this technique is superior to other methods.