

# **New Directions in High-Order Adaptive Methods for Computational Fluid Dynamics**

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Symposium in Honor of Antony Jameson's 80<sup>th</sup> birthday

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# Introduction

## Complex CFD simulations made possible by

- Increasing computational power
- Improvements in numerical algorithms

## New liability: ensuring accuracy of computations

- Management by expert practitioners is not feasible for increasingly-complex flow fields
- Reliance on best-practice guidelines is an open-loop solution: numerical error is unchecked for novel configurations
- Output calculations are not yet sufficiently robust, even on relatively standard simulations

⇒ **research into adaptive methods that can “close the loop”**

# Successful adaptive methods require

## 1. Accurate error estimation

- How much error is present?
- Where is it coming from?

⇒ **Approach:** *adjoint-weighted residual*

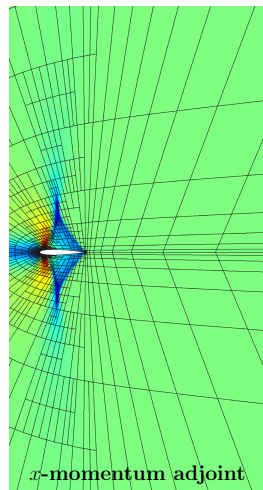
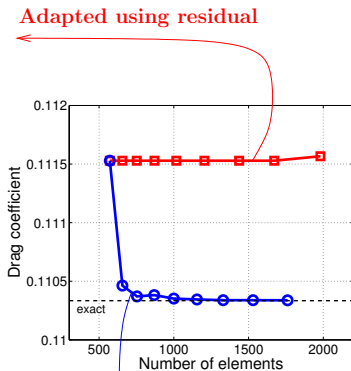
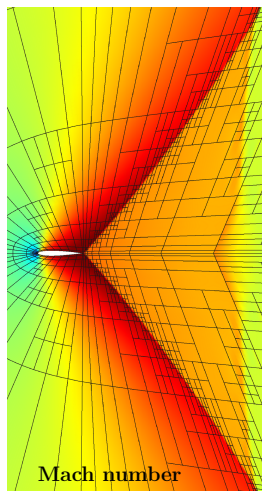
## 2. Efficient adaptation

- How should degrees of freedom be introduced?
- Can scheme approximation properties be improved?
- Is the scheme itself adequate?

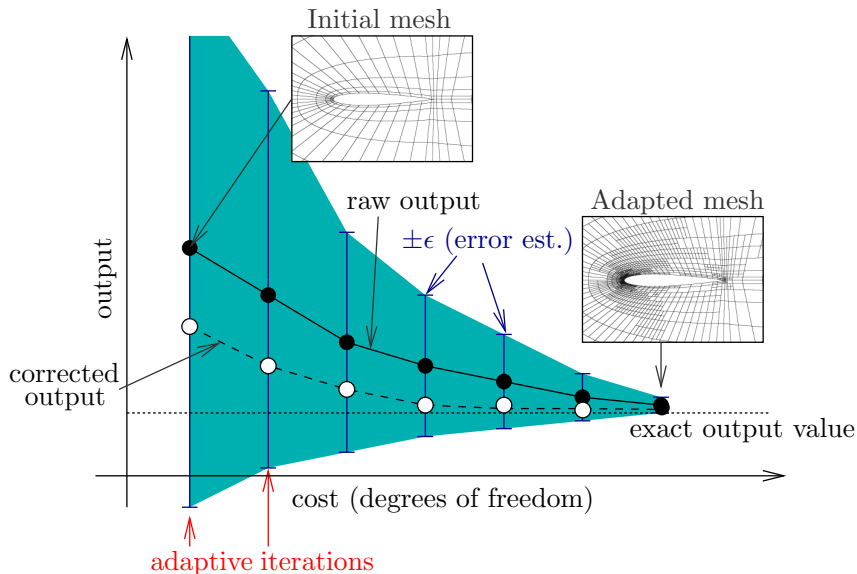
⇒ **Approach:** *a diverse set of adaptation strategies*

# Why not just adapt “obvious” regions?

Fishtail shock in  $M_\infty = 0.95$  inviscid flow over a NACA 0012 airfoil

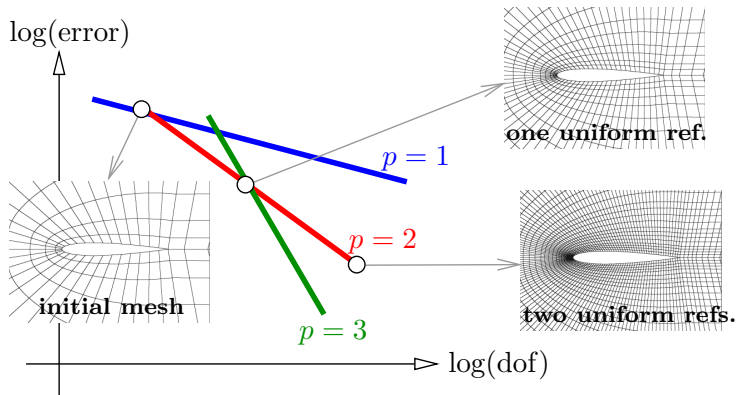


# A typical output-adaptive result



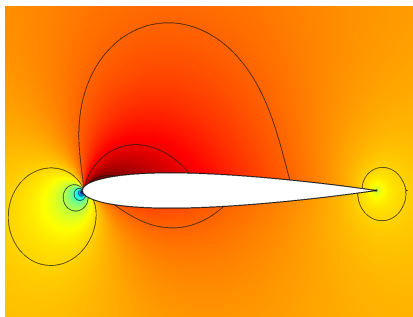
# Adaptive mechanics: the role of high order

- High-order methods: errors converge faster than 2<sup>nd</sup>-order
- Typically choose high-order methods for “smooth” problems, where we expect to see convergence plots that look like:

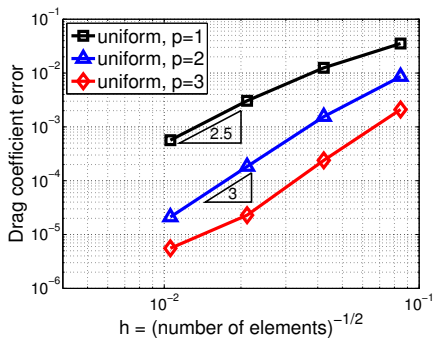


# Can aero applications benefit from high order?

- Question considered by recent high-order CFD workshops
- Aerospace applications usually have both smooth and singular features (shocks, trailing edges)
- Singularities can limit observed rates



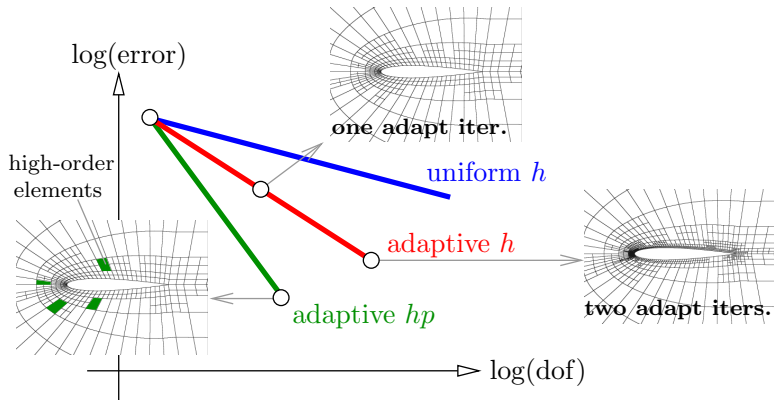
Mach numbers (0–0.7), Euler flow  
over a NACA 0012 airfoil



Drag error convergence

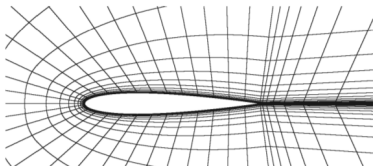
# High-order in mesh adaptation

- $h$ -Adaptation can isolate singularities with small elements
- In many high-order methods, local  $p$ -enrichment is possible
- High-order just becomes another refinement tool for efficiently improving accuracy

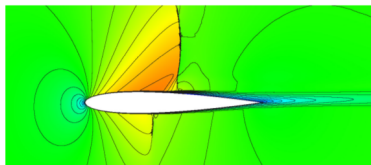


# We have done many $h/p$ adaptive runs

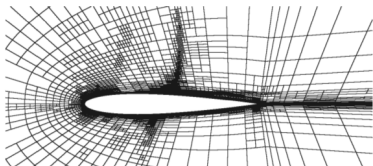
## Transonic RANS flow over a NACA 0012



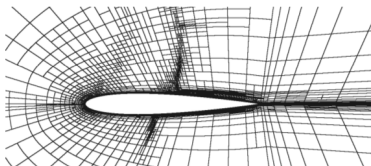
23.1: Initial mesh (1740 elements)



23.2: Mach number contours

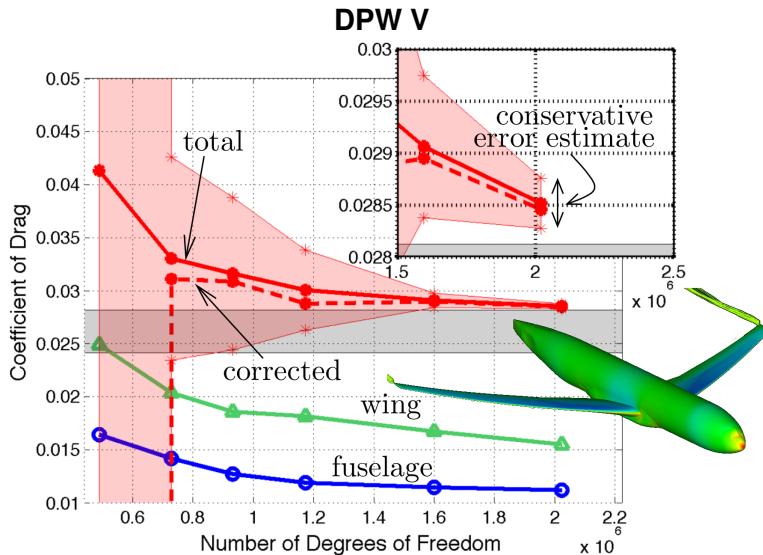


23.3: 6<sup>th</sup> adapted mesh, isotropic (8,736 elements)



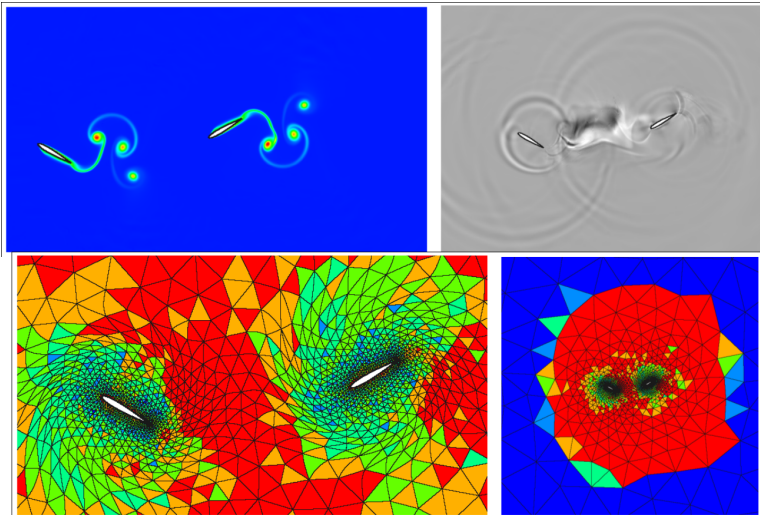
23.4: 10<sup>th</sup> adapted mesh, anisotropic (4,816 elements)

# We have done many $h/p$ adaptive runs



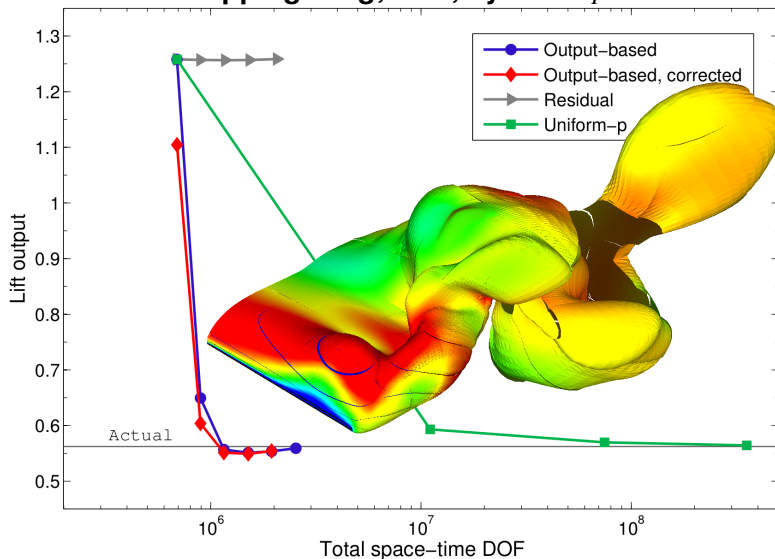
# We have done many $h/p$ adaptive runs

Staggered pitching/plunging airfoils; ALE, dynamic  $p$



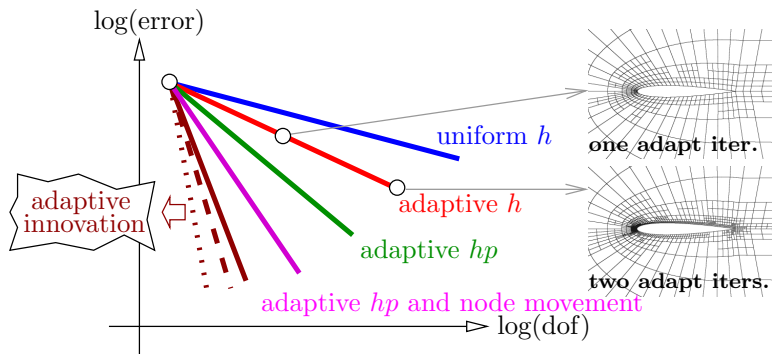
# We have done many $h/p$ adaptive runs

## Flapping wing; ALE, dynamic $p$



# Novel adaptive strategies (for FEM)

- **Automatic hp**: choice driven by error indicator
- **Node movement**: warp elements to improve approximation
- **Trial space tailoring**: better than polynomials
- **Test space optimization**: not just Galerkin
- ...



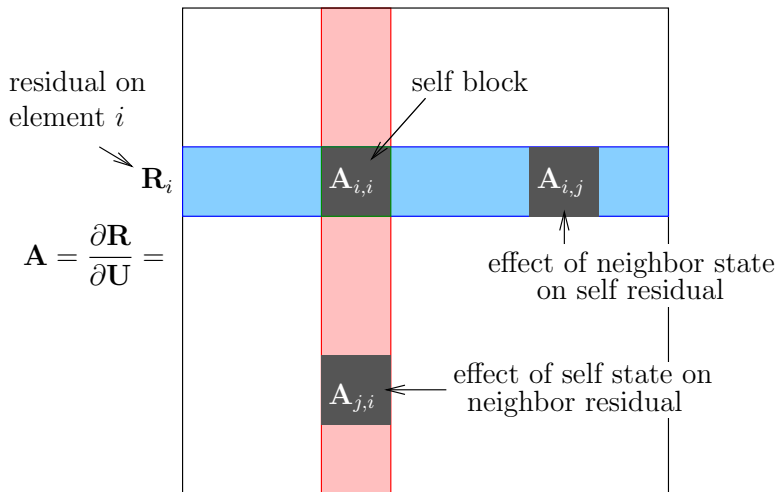
# Trial space tailoring

- Polynomial approximation spaces are popular

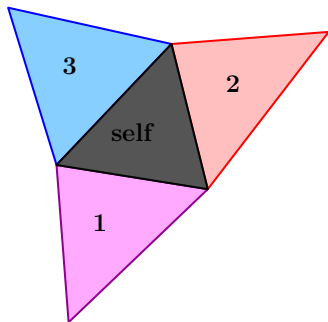
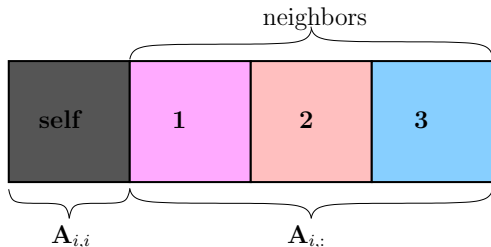
$$u(\vec{x}) = \sum_j U_j \phi_j(\vec{x}), \quad \phi_j(\vec{x}) \in \mathcal{P}^p$$

- But they are not always the most efficient
- In FEM, we can tailor  $\{\phi_j(\vec{x})\}$  to each element
- One application:  $p$ -multigrid coarse spaces
  - Typically  $\mathcal{P}^{p-1}, \mathcal{P}^{p-2}, \dots$
  - We consider *arbitrary* subspaces  $\mathcal{V}_H \subset \mathcal{P}^p$
  - Similar to algebraic multigrid, locally on each element
- First approach: linear  $p$ -multigrid solver acceleration

# A closer look at the residual Jacobian matrix



# Compress the $i^{\text{th}}$ block row



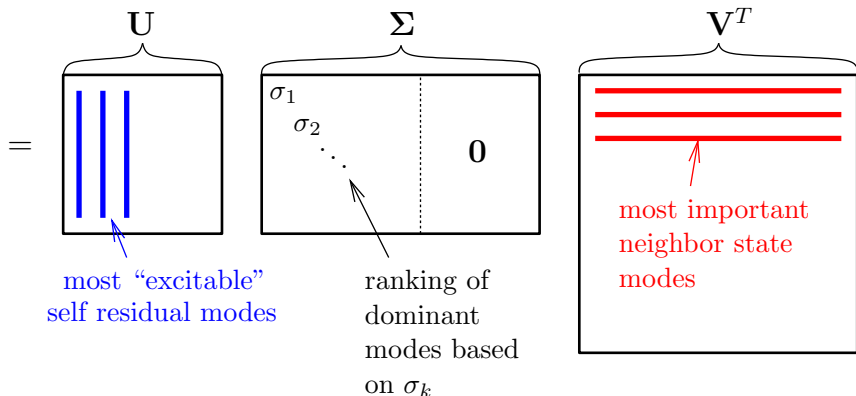
## Observations:

- Neighbor state perturbations may excite only some residual modes on element  $i$
- These modes are typically not excited to the same degree
- Not all neighbor state perturbations affect residuals in element  $i$

# Apply the SVD

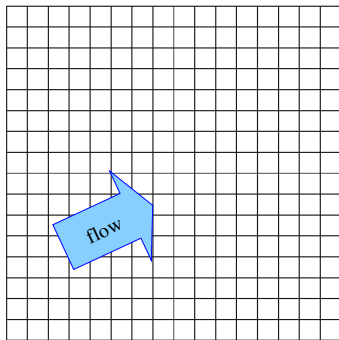
Take the singular value decomposition (SVD) of  $\mathbf{A}_{i,:}$ :

$$\text{SVD}(\mathbf{A}_{i,:}) = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$$



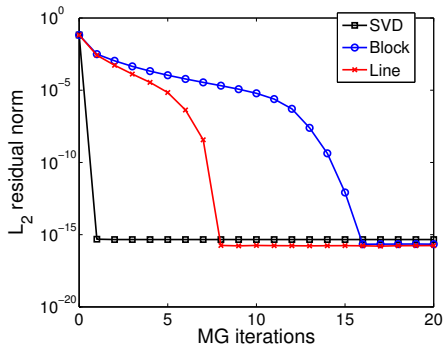
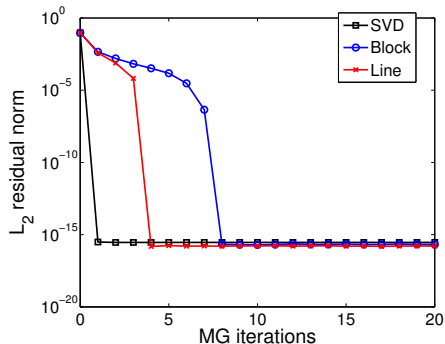
$p$ -Multigrid: coarse-space test/trial functions from  $\mathbf{U}$  and  $\mathbf{A}_{ii}^{-1}\mathbf{U}$ .

## 2D advection example



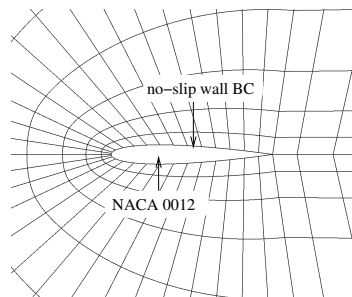
- 2D linear advection (at  $25^\circ$ )
  - Quadrilateral meshes
  - Constant inflow state BC
  - Random initial condition
  - DG Lagrange basis
  - Fine space:  $p_h = 3 \Rightarrow r_h = 16$
  - Coarse space:  $p_H = 2 \Rightarrow r_H = 9$
  - Two-grid cycle,  $\nu_{\text{pre}} = \nu_{\text{post}} = 2$
- 
- We compare convergence of MG for block, line, SVD smoothers
  - Coarse space size of SVD MG is the same size as in block/line

# 2D advection: convergence rates



- One MG iteration is an exact solve with the SVD smoother
- Same result at higher orders  $p_h$

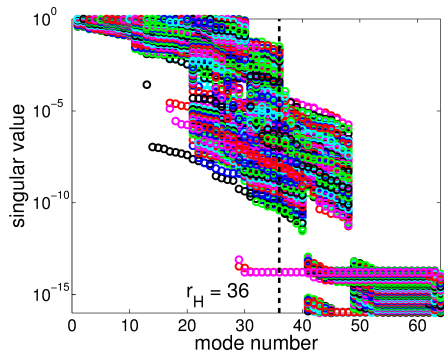
# 2D linearized Euler example



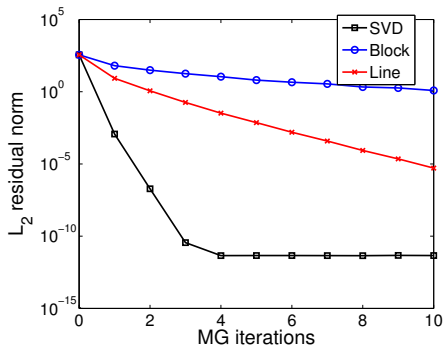
- 2D linearized Euler  
( $M = 0.5, \alpha = 2^\circ$ )
- Quadrilateral curved meshes
- Random initial condition
- DG Lagrange basis
- Fine space:  $p_h = 3 \Rightarrow r_h = 64$
- Coarse space:  $p_H = 2 \Rightarrow r_H = 36$
- Two-grid cycle

- We compare convergence of MG for block, line, SVD smoothers
- Coarse space size of SVD MG is the same size as in block/line

# 2D linearized Euler: singular values, convergence



Singular values



Residual convergence

- (# nonzero singular values)  $> r_H$
- Now need more than one MG-SVD iteration
- Significantly fewer MG iterations compared to block or line

# Another way to adapt

So far

- Standard “general-purpose” numerical scheme (DG/HDG)
- Output-based *mesh* adaptation

Another direction

- Optimize the numerical scheme itself
- Specifically, we optimize the FEM *test space*

$$\int_{\Omega} \mathbf{v}^T \mathbf{r}(\mathbf{u}) \, d\Omega = 0 \quad \mathbf{u} \in \mathcal{P}^p, \quad \mathbf{v} \in \mathcal{P}^{p_{\text{test}}}$$

Results in a Petrov-Galerkin method

- Our focus: boundary accuracy for engineering outputs
- Applied to HDG  $\rightarrow$  **HDPG**

# Optimal test functions

## Key idea

Choose test functions that make FEM weighted residual look like an error minimization equation

**Example:** 1D advection on one element

$$Lu \equiv a \frac{\partial u}{\partial x} = f, \quad a > 0, \quad u(x_L) = u_L$$

Discrete weight. residual:  $\int_{\Omega} v_h(Lu_h - f) dx = 0$

- exact weight. residual:  $\int_{\Omega} v_h L(u_h - u) dx = 0$

Move  $L$  to test functions:

$$\int_{\Omega} L^* v_h(u_h - u) dx + v_h a(u_h - u)|_{x_R} = 0$$

# 1D HDPG example (ctd)

What do we want from the solution? A reasonable error norm is

$$\|e\|^2 = \int_{\Omega} (u_h - u)^2 dx + w_R (u_h - u)^2 \Big|_{x_R}$$

For  $u_h = \sum_j U_j \phi_j$ , the error minimization equation reads

$$\frac{\partial \|e\|^2}{\partial U_j} = 0 \quad \rightarrow \quad \int_{\Omega} \phi_j (u_h - u) dx + w_R \phi_j (u_h - u) \Big|_{x_R} = 0$$

We can make our weighted residual look like this if the test functions,  $v_j$ , satisfy an **adjoint equation**:

$$\begin{aligned} L^* v_j &= \phi_j \\ v_j \Big|_{x_R} &= \frac{w_R}{a} \phi_j \Big|_{x_R} \end{aligned}$$

## Example: 1D advection (ctd.)

Such test functions are adjoints for the outputs,

$$J_j = \int_{\Omega} \phi_j u \, dx + w_R \phi_j u \Big|_{x_R}$$

Our FEM weighted residual is then an **adjoint-weighted residual**,

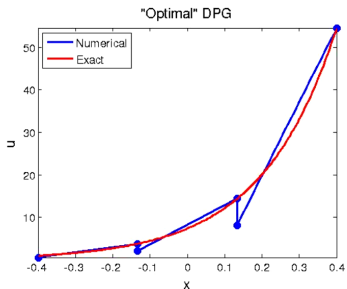
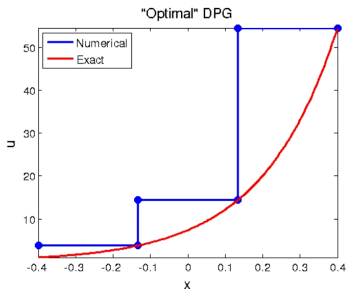
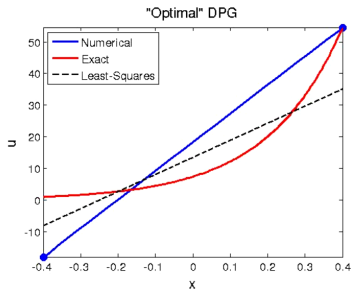
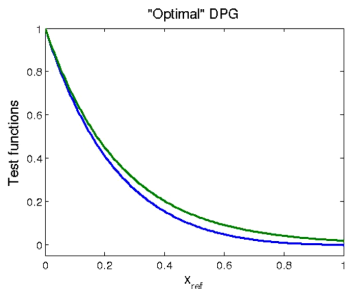
$$\int_{\Omega} v_i (Lu_h - f) \, dx = 0 \quad \rightarrow \quad J_j(u_h) - J_j(u) = 0$$

By choosing large  $w_R$  we can emphasize **boundary accuracy**

Note:

- $v_j$  must lie in an enriched space (else standard Galerkin)
- For multiple elements we need to localize  $v_j$  calculation

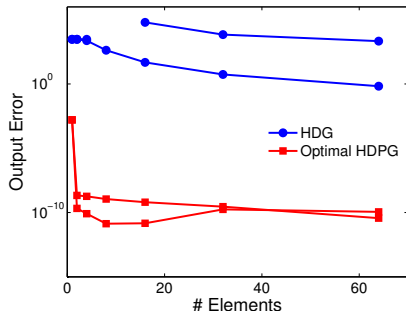
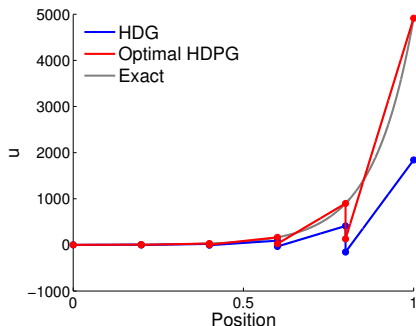
# 1D scalar result: effect of $w_R$



# 1D Advection with source

$$a \frac{\partial u}{\partial x} - cu = 0, \quad u|_{x_L} = 1$$

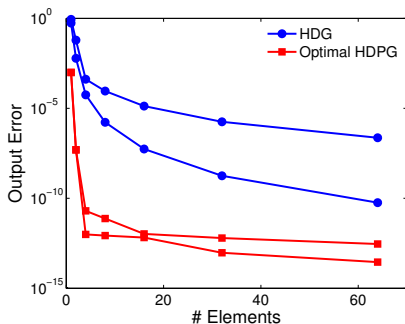
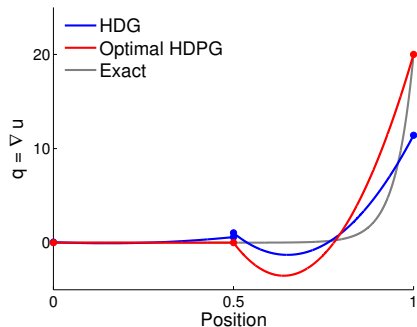
- $a = 1$  and  $c = 8$
- For boundary accuracy, use  $w_R = 1 \times 10^{10}$  and  $p_{test} = 10$
- $p = 0$  and  $p = 1$  convergence is shown



# 1D Advection-diffusion

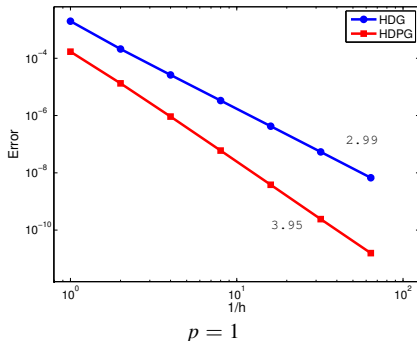
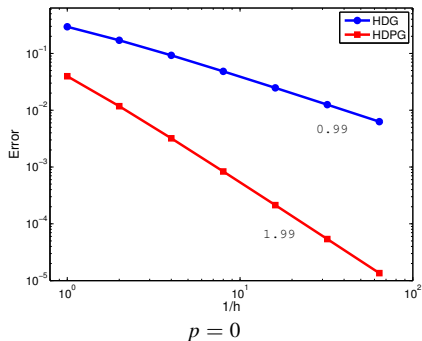
$$a \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} = 0 \quad u|_{x_L} = 0, \quad u|_{x_R} = 1$$

- $a/\nu = 20$
- $w_L, w_R = 1 \times 10^{12}$  and  $p_{test} = 10$
- $p = 1$  and  $p = 2$  convergence is shown



# Nonlinear systems: Euler equations

- Supersonic Euler, inflow Mach = 2, Dirichlet BCs on left
- Source term  $S^T = [0.4\rho^2 \quad 0.7(\rho u)^2 \quad 0.1(\rho H)^2]$
- $p_{test} = p + 1$ , boundary weights =  $1 \times 10^7$



- $2p + 2$  rates are obtained
- Subsonic cases show similar behavior, though with rates of  $\approx 2p + 1.75$

# Extension to multiple dimensions

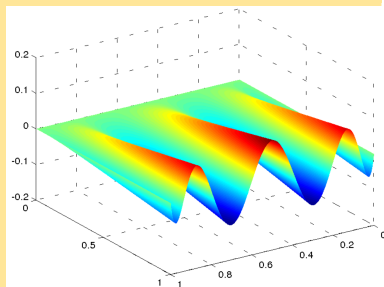
## Outputs for test function adjoint calculations

$$J^{k,i} = \int_{\Omega} u_k \phi_{k,i} d\Omega + \int_{\partial\Omega} \sum_{s=1}^{sr} \left[ w_s \frac{\partial \hat{F}_s}{\partial u_{k,h}} \hat{F}_s(\mathbf{u}, \vec{\mathbf{q}}) \right] \phi_{k,i} ds$$

Similarly for components of  $\vec{\mathbf{q}}$  in HDG

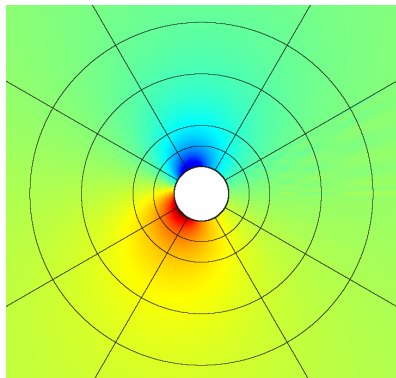
## Trial space enrichment

- Need accurate test fcn *and* fluxes
- In 2D, the **trial basis needs to be enriched** to represent the fluxes
- Enrich the trial space with Lobatto edge functions of order  $p_B$
- Inside elements:  $p_I < p_B$
- Total *DOF* scales linearly with  $p_B$

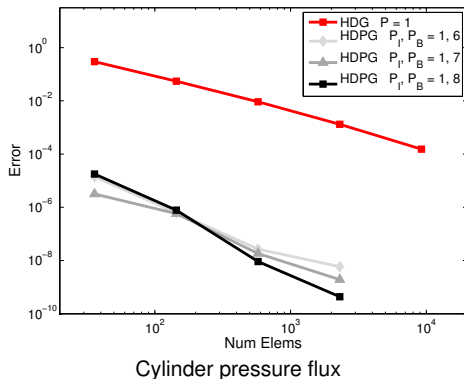


# HDG for flow over a cylinder: 2D linearized Euler

- Mach number of 0.3
- $p_{test} = p_b$ , boundary weights =  $1 \times 10^{10}$
- HDPG interior order kept at  $p_I = 1$ , while  $p_B$  is enriched



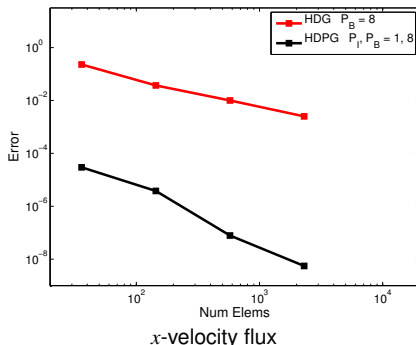
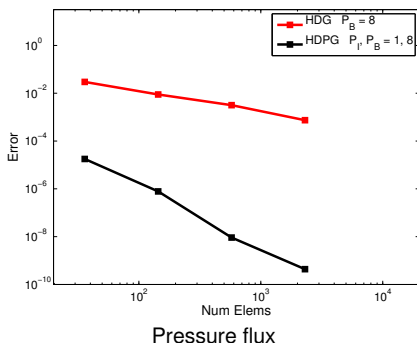
Pressure contours



Cylinder pressure flux

# HDG for flow over a cylinder: 2D linearized Euler

- Here, we compare standard HDG with  $p_B = 8$  edge enrichment to HDPG with the same enrichment
- **The only difference is the test space**
- We see that HDPG achieves much better accuracy



# Acknowledgements

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— Thank you —