Local Fourier-spectral filtering for non-linear stabilization of high-order Flux Reconstruction schemes

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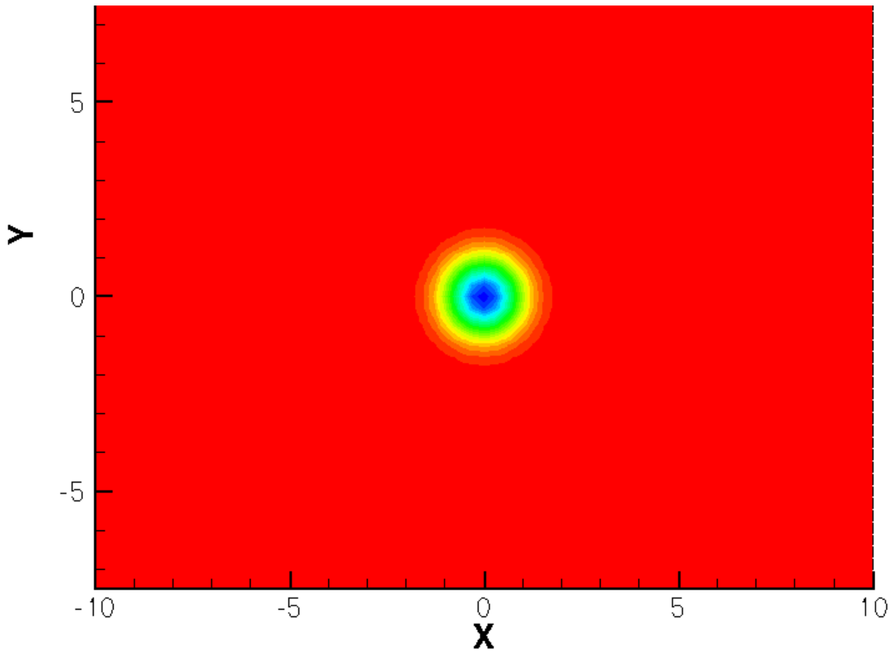
Outline

1. Why High-Order methods?
2. The Flux Reconstruction Scheme
3. About HiFiLES
4. Local Fourier-spectral Filters
5. Results
6. Conclusion

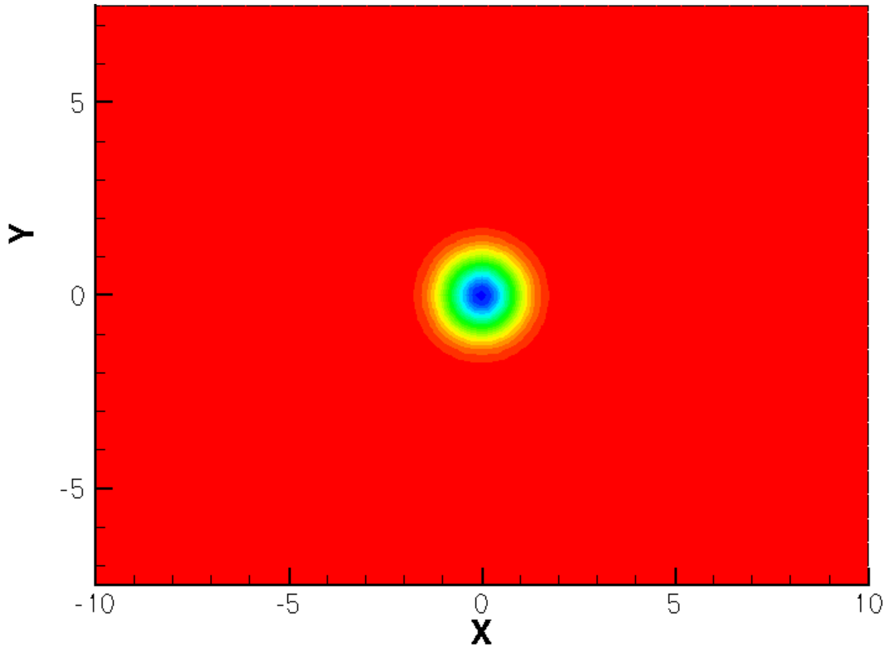
Why High-Order Methods?

Example: Linear Advection of Isentropic Vortex

2nd Order



4th Order



equal # DoF

Outline

1. Why High-Order methods
2. The Flux Reconstruction Scheme
   a) local matrix-vector multiplications
   b) unstructured, parallelizable
3. About HiFiLES
4. Local Fourier-spectral Filters

Flux Reconstruction

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0 \qquad \Omega_n = [x_n, x_{n+1})$$

$$\frac{\partial \hat{u}}{\partial t} + \frac{1}{J_n}\frac{\partial \hat{f}}{\partial \xi} = 0 \qquad \Omega_s = [-1, 1)$$

$$\hat{u}^\delta = \sum_{i=1}^{N_s} \hat{u}_i^\delta l_i(\xi) \qquad \hat{f}^\delta = \sum_{i=1}^{N_s} \hat{f}_i^\delta l_i(\xi)$$

Flux Reconstruction

$$\hat{f}^C = \hat{f}^\delta + (\hat{f}_L^{\delta I} - \hat{f}_L^\delta) g_L + (\hat{f}_R^{\delta I} - \hat{f}_R^\delta) g_R$$

$$g_L(-1) = 1 \, , \; g_L(1) = 0$$

$$g_R(-1) = 0 \, , \; g_R(1) = 1$$

$$\frac{d\hat{u}_i^\delta}{dt} = -\frac{1}{J_n} \frac{\partial \hat{f}^C}{\partial \xi}(\xi_i)$$

Element-wise matrix-vector multiplications advance the solution!

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High Fidelity Large Eddy Simulation

HiFiLES

- High-Order via Flux Reconstruction scheme
- RANS/LES
- GPU/CPU scalable
- Unstructured grids
- 4th order explicit time-stepping

hifiles.stanford.edu

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1. Why High-Order methods
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4. Local Fourier-spectral Filters
    a) stability of FR
    b) non-linear stability can be achieved
    c) equivalence with artificial dissipation
    d) filter design

Stability of Flux Reconstruction

- Energy Stable Flux Reconstruction:
  Vincent et al. (2010) showed that for linear fluxes, FR can be made stable
- But for non-linear fluxes, aliasing becomes a problem:

$$\frac{1}{2}\frac{d}{dt}\left|u^{\delta}\right|^2_{\Omega^{\delta},(P,2)} \leq \sum_{n=0}^{N-1} \int_{\Omega_n} \left(f_n^{\delta} - f\right) \frac{\partial u_n^{\delta}}{\partial x} \mathrm{d}x$$

$\left|u^{\delta}\right|_{\Omega^{\delta},(P,2)}$ is a broken Sobolev-type norm

Jameson et al., 2011

Local Fourier-spectral Filters

Wish to stabilize scheme by adding diffusion

$$\frac{\partial u^\delta}{\partial t} + \frac{\partial I_P(f)}{\partial x} = \sum_{s=1}^{\infty} (-1)^{s+1} \epsilon_s \frac{\partial^{2s} u^\delta}{\partial x^{2s}}$$

Asthana et al., 2014

Local Fourier-spectral Filters

In the case of $u(x), f(u(x)) \in H^1(\Omega^\delta)$

$$\frac{1}{2}\frac{d}{dt}|u^\delta|^2_{\Omega^\delta,(P,2)} \leq C_1(1+C_2)h \left\|\frac{\partial f}{\partial x}\right\|_{\Omega^\delta,0} \left\|\frac{\partial u}{\partial x}\right\|_{\Omega^\delta,0} - \sum_{s=1}^{\infty} \epsilon_s \left\|\frac{\partial^s u^\delta}{\partial x^s}\right\|^2_{\Omega^\delta,0}$$

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Local Fourier-spectral Filters

$$\epsilon_1 > 0, \epsilon_j \geq 0 \; for \; j = 2, \ldots, P$$

$$\epsilon_1 \propto \frac{h^a (P+1)^b}{\Delta t_{CFL}}, \quad a < \frac{5}{2}, b > -1$$

For $h < h_{crit}(P)$, $\frac{1}{2}\frac{d}{dt}|u^\delta|^2_{\Omega^\delta,(P,2)} \leq 0$

⇒ There *is* hope!

Asthana et al., 2014

Local Fourier-spectral Filters

Can pose addition of hyperviscosity as a Fourier filtering operation!

$$G(k) = 1 - \sum_{s=1}^{\infty} \alpha_s k^{2s} \quad ; \quad \alpha_s = \epsilon_s \Delta t$$

Can implement Fourier filtering via convolution

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Local Fourier-spectral Filters

$$v(\boldsymbol{\xi}) = \sum_{i=1}^{N_s} v_i \phi_i(\boldsymbol{\xi})$$

$$\begin{aligned}
\bar{v}_j &= (G * v)(\boldsymbol{\xi}_j) \\
&= \int_{\mathbb{R}^d} G(\boldsymbol{\xi}_j - \boldsymbol{\xi}) v(\boldsymbol{\xi}) d\boldsymbol{\xi} \\
&= \int_{\mathbb{R}^d} G(\boldsymbol{\xi}_j - \boldsymbol{\xi}) \left(\sum_{i=1}^{N_s} v_i \phi_i(\boldsymbol{\xi}) \right) d\boldsymbol{\xi} \\
&= \sum_{i=1}^{N_s} v_i \int_{\mathbb{R}^d} G(\boldsymbol{\xi}_j - \boldsymbol{\xi}) \phi_i(\boldsymbol{\xi}) d\boldsymbol{\xi}
\end{aligned}$$

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Local Fourier-spectral Filters

$$\begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \\ \vdots \\ \bar{v}_{N_s} \end{bmatrix} =$$

$$\begin{bmatrix} \int_{\mathbb{R}^d} G(\xi_1 - \xi)\phi_1(\xi)d\xi & \int_{\mathbb{R}^d} G(\xi_1 - \xi)\phi_2(\xi)d\xi & \cdots & \int_{\mathbb{R}^d} G(\xi_1 - \xi)\phi_{N_s}(\xi)d\xi \\ \int_{\mathbb{R}^d} G(\xi_2 - \xi)\phi_1(\xi)d\xi & \int_{\mathbb{R}^d} G(\xi_2 - \xi)\phi_2(\xi)d\xi & \cdots & \int_{\mathbb{R}^d} G(\xi_2 - \xi)\phi_{N_s}(\xi)d\xi \\ \vdots & \vdots & \ddots & \vdots \\ \int_{\mathbb{R}^d} G(\xi_{N_s} - \xi)\phi_1(\xi)d\xi & \int_{\mathbb{R}^d} G(\xi_{N_s} - \xi)\phi_2(\xi)d\xi & \cdots & \int_{\mathbb{R}^d} G(\xi_{N_s} - \xi)\phi_{N_s}(\xi)d\xi \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{N_s} \end{bmatrix}$$

$$\vec{\bar{v}} = \tilde{G}\vec{v}$$

Filtering in each element becomes a matrix multiplication!

Asthana et al., 2014

Local Fourier-spectral Filters

$$\int_{\mathbb{R}^d} G(\boldsymbol{\xi}_j - \boldsymbol{\xi})\phi_i(\boldsymbol{\xi})d\boldsymbol{\xi} =$$

$$\int_{\Omega_n} G(\boldsymbol{\xi}_j - \boldsymbol{\xi})\phi_i(\boldsymbol{\xi})d\boldsymbol{\xi} + \int_{\mathbb{R}^d \setminus \Omega_n} G(\boldsymbol{\xi}_j - \boldsymbol{\xi})\phi_i(\boldsymbol{\xi})d\boldsymbol{\xi}$$

Basis function needs to be defined outside of Ω_n

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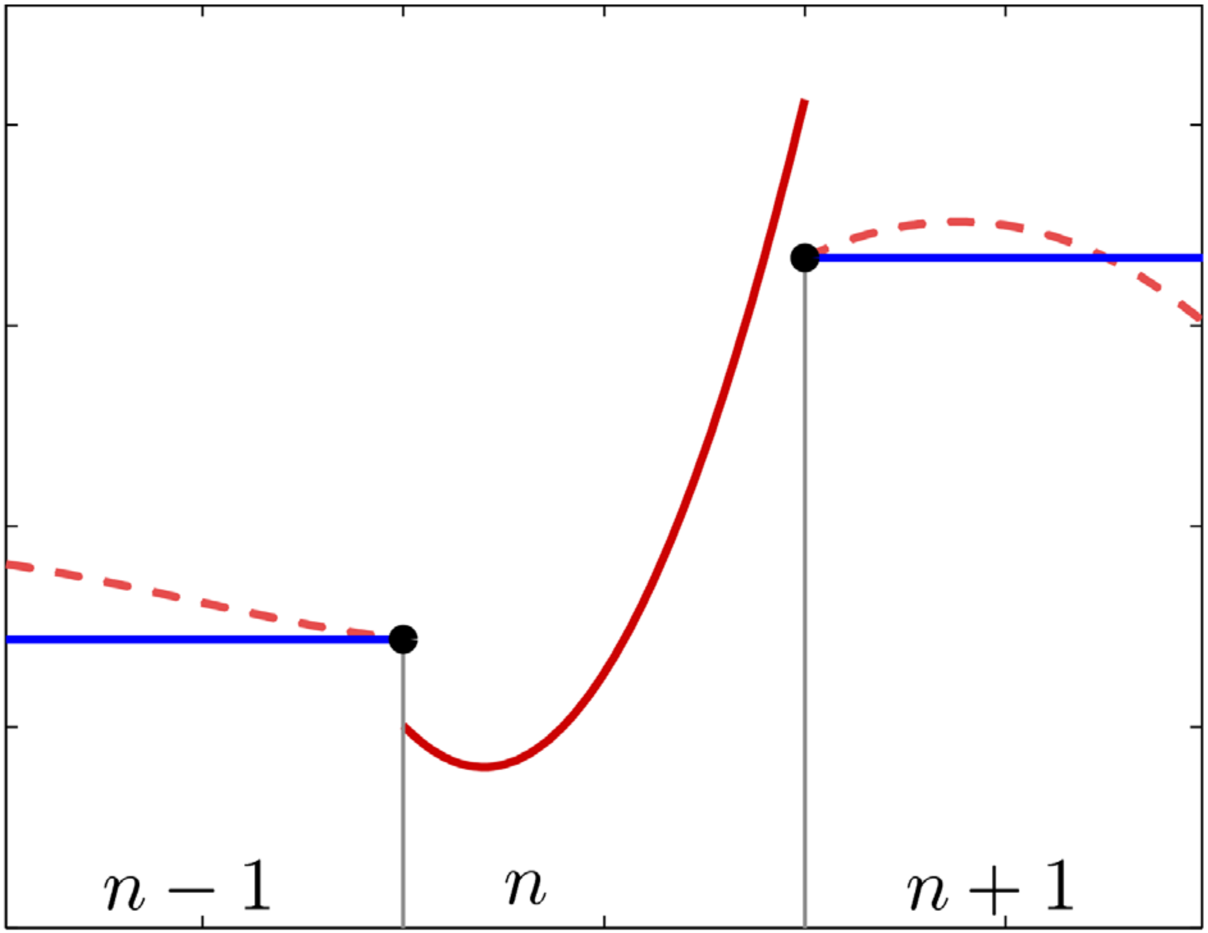
Local Fourier-spectral Filters

Possible choice in 1D:



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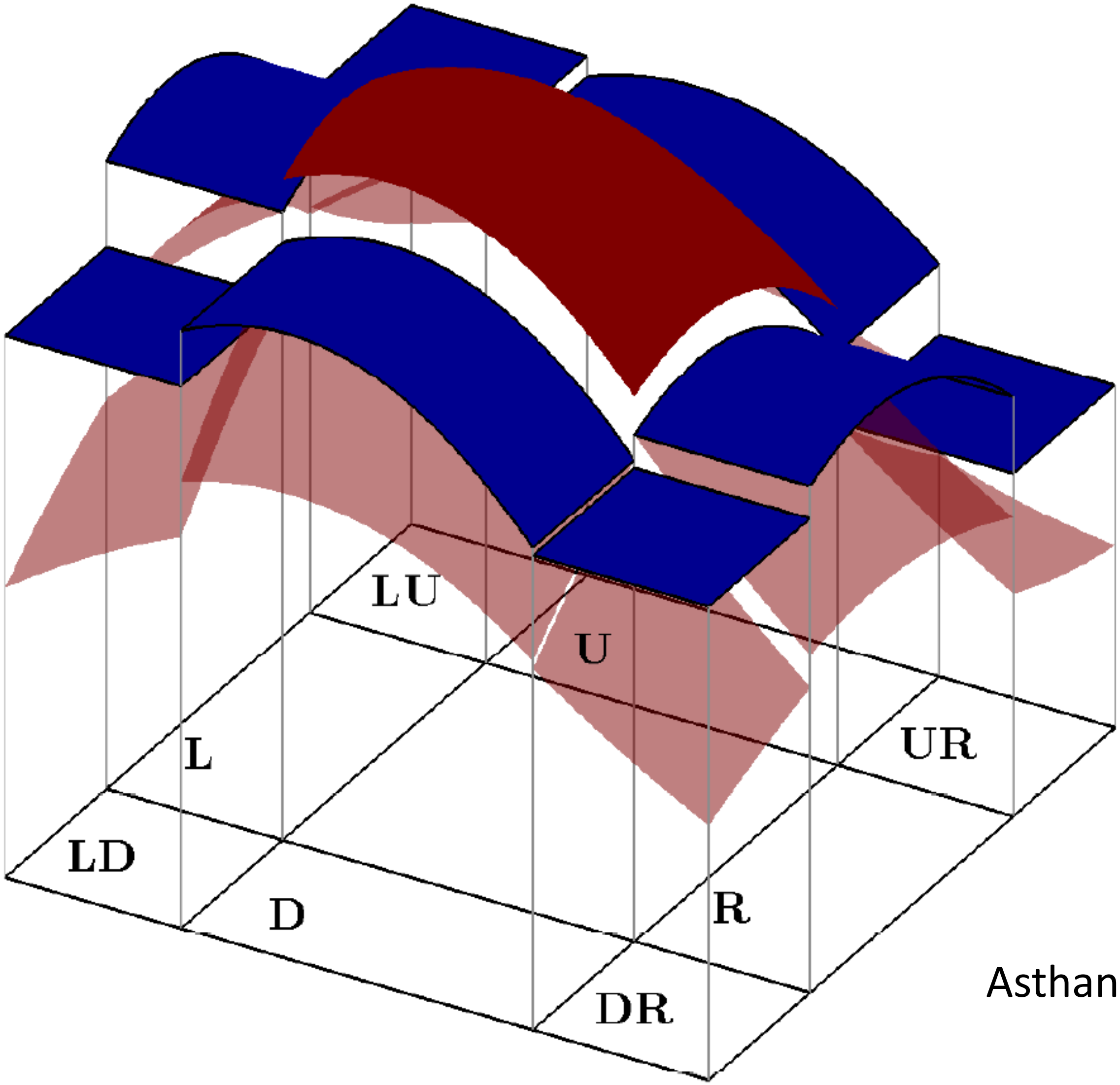
Local Fourier-spectral Filters

Possible choice in 2D:



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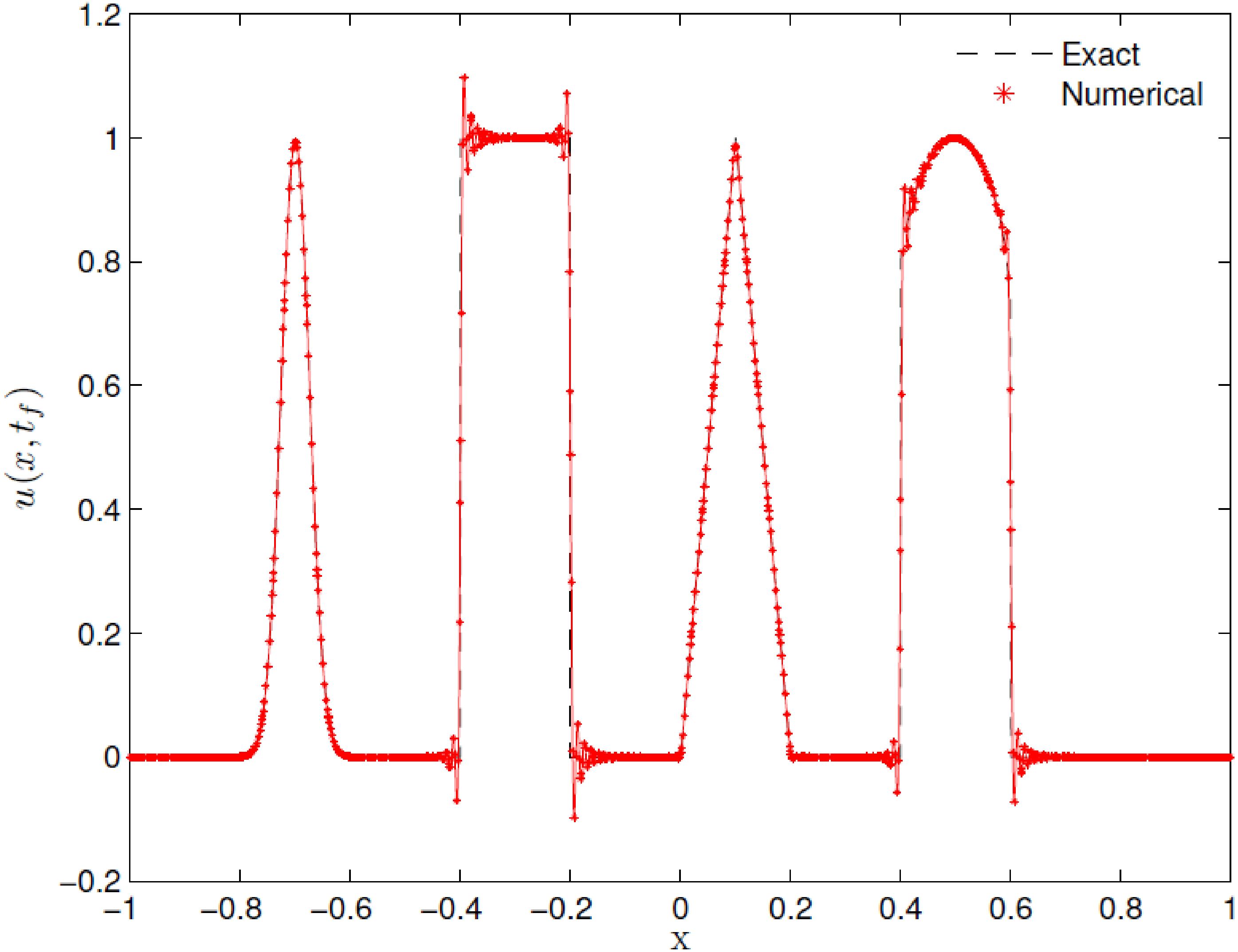
4. Local Fourier-spectral Filters
5. Results
    1. concentration sensor
    2. 1D
    3. 2D

Results

Unfiltered FR; advection; P=8;N=100



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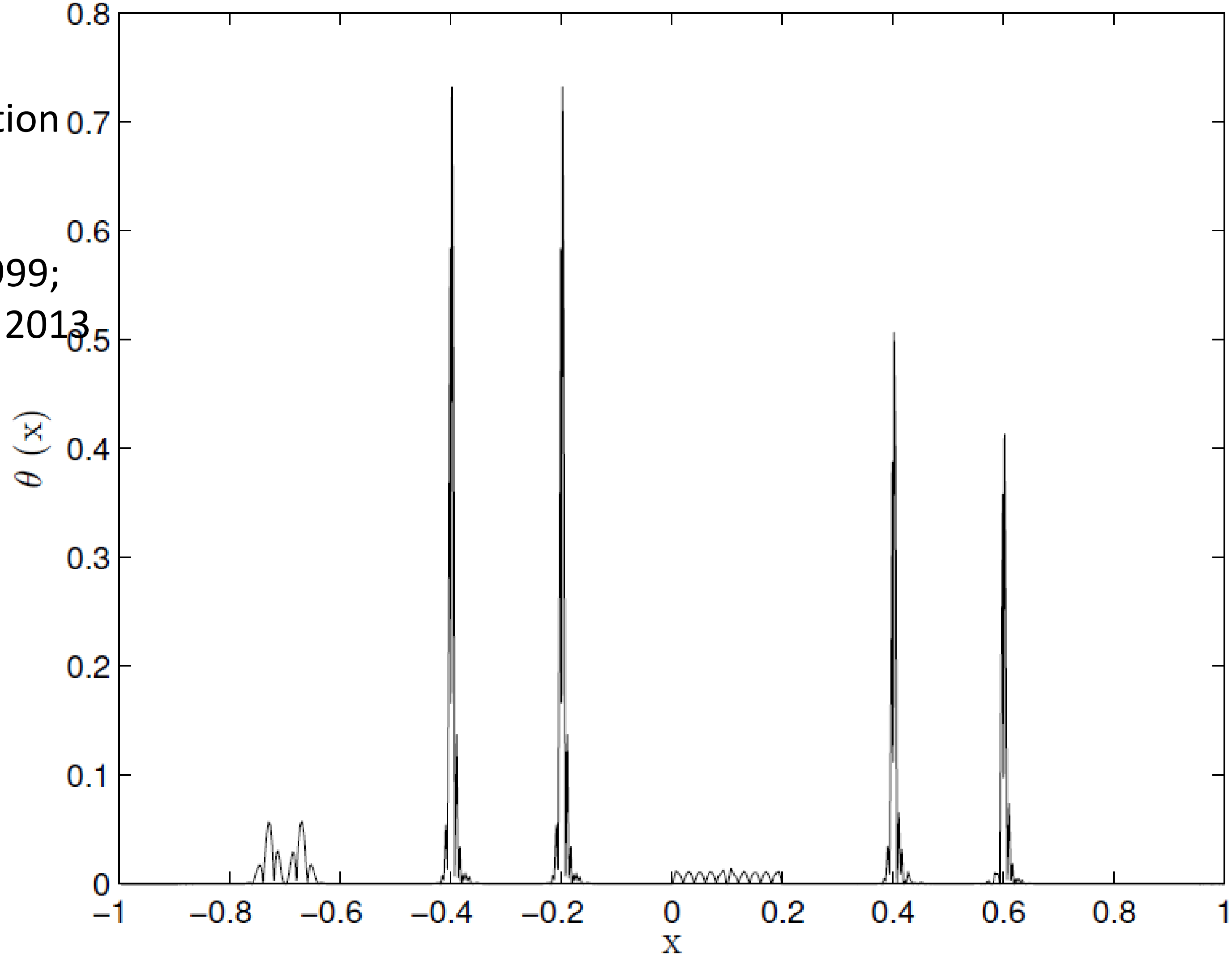
Results

Sensor; advection; P=8;N=100

Sensor: Concentration sensor by Gelb and Tadmor, 1999; Sheshadri, 2013



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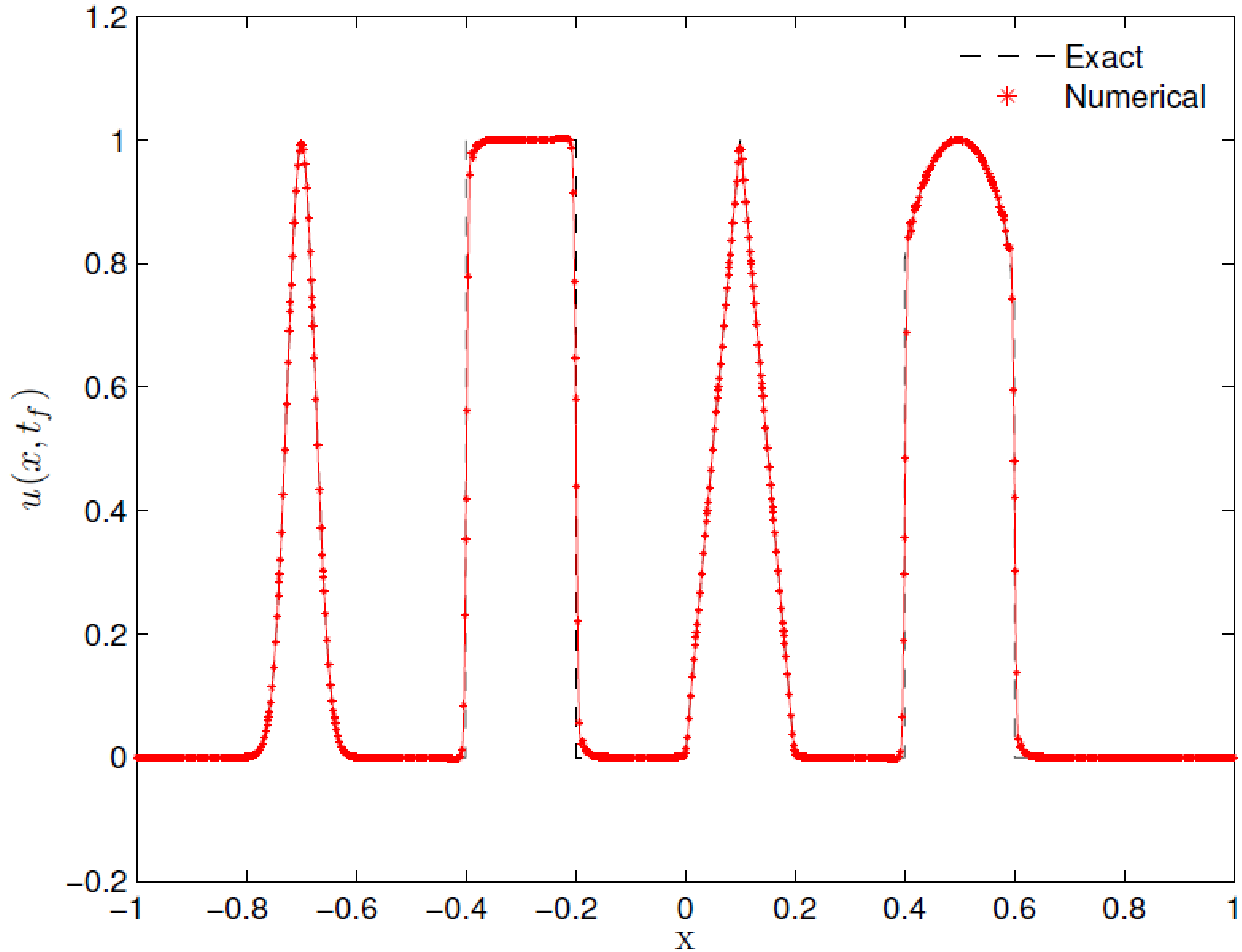
Results

Filtered FR; box filter; advection; P=8;N=100



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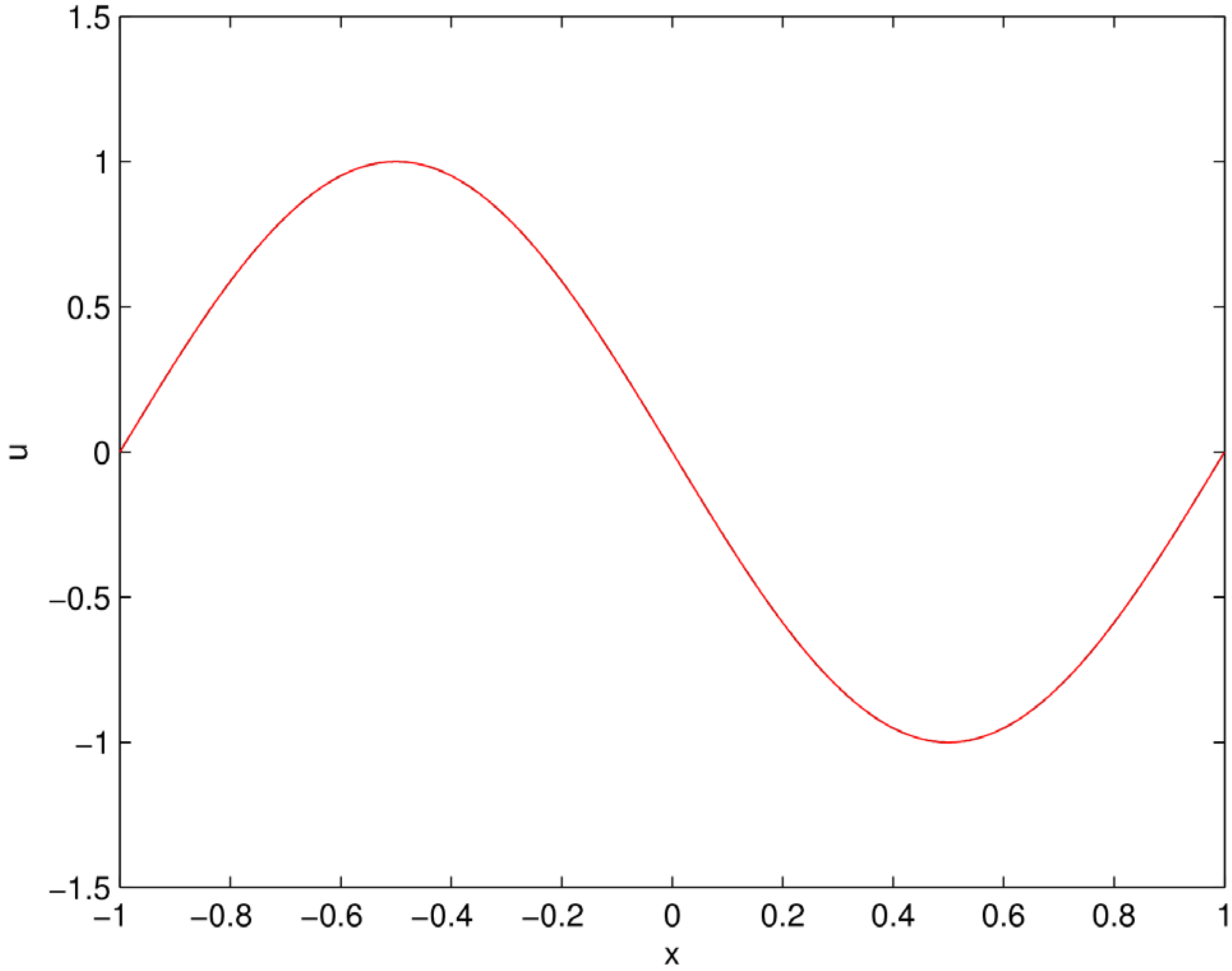
Results

Burgers; P=8;N=40; initial condition



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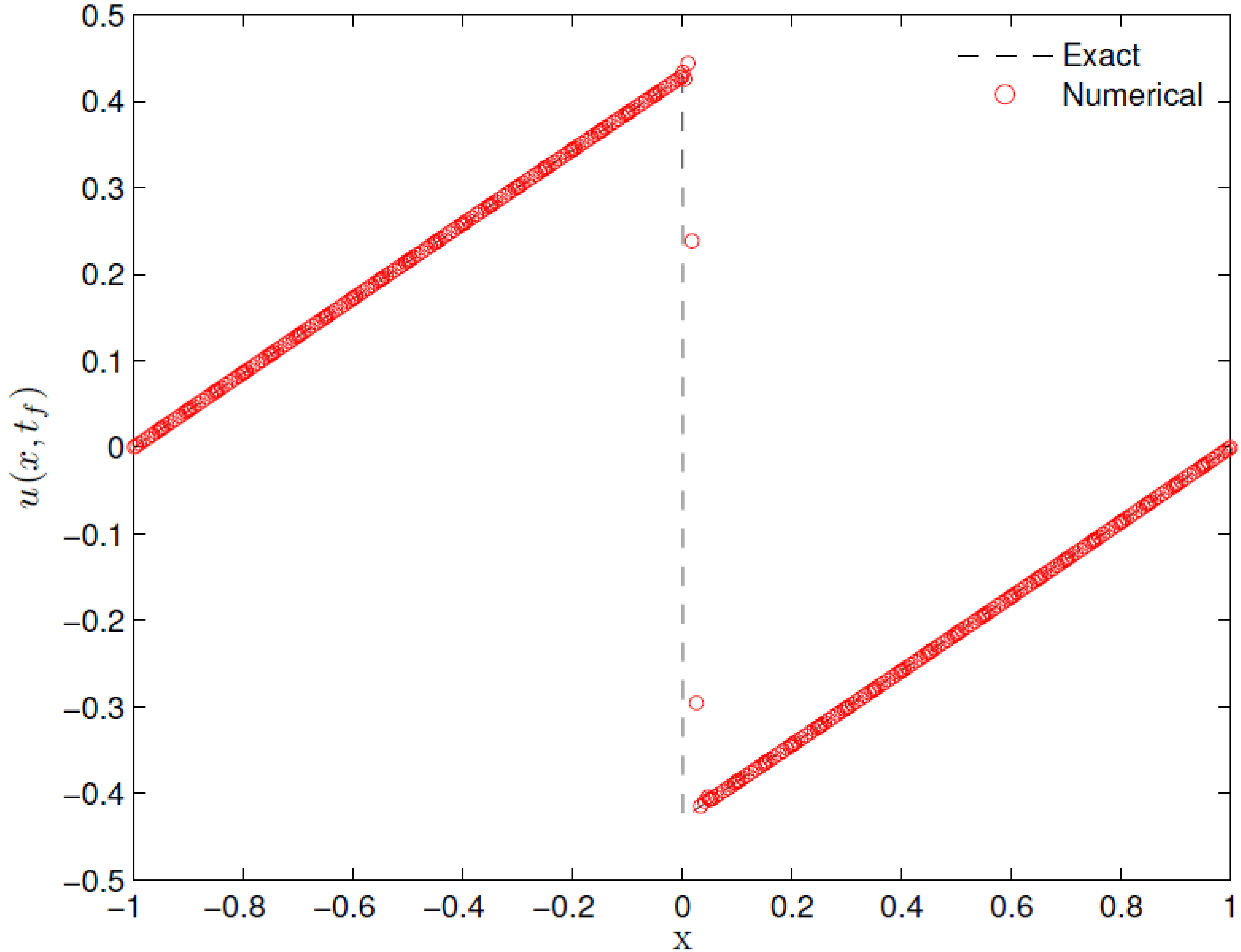
Results

Burgers; P=8;N=40; impossible without filtering



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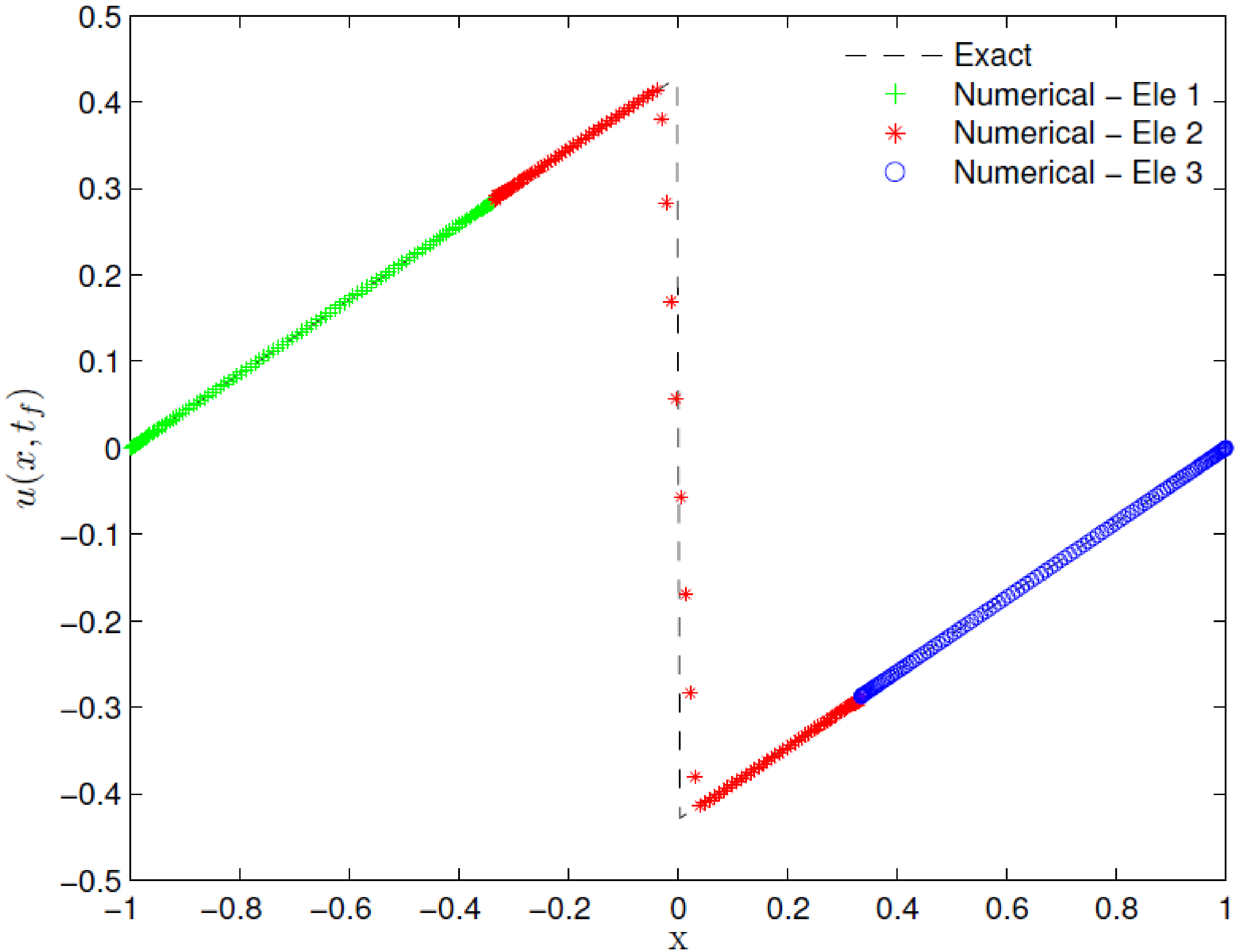
Results

Box filter; Burgers; P=119; N=3



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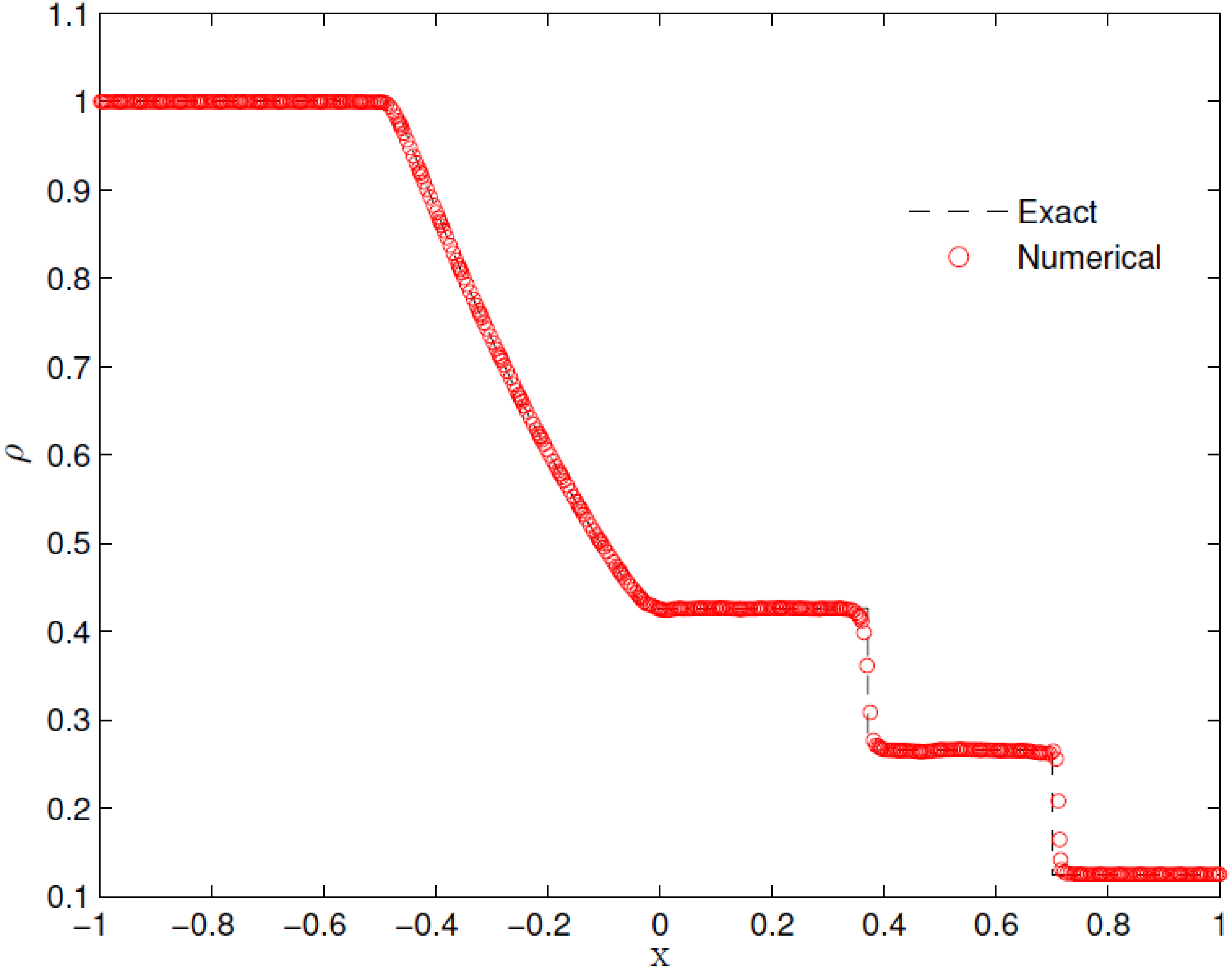
Results

Box filter; Sod's shock tube; P=8;N=56



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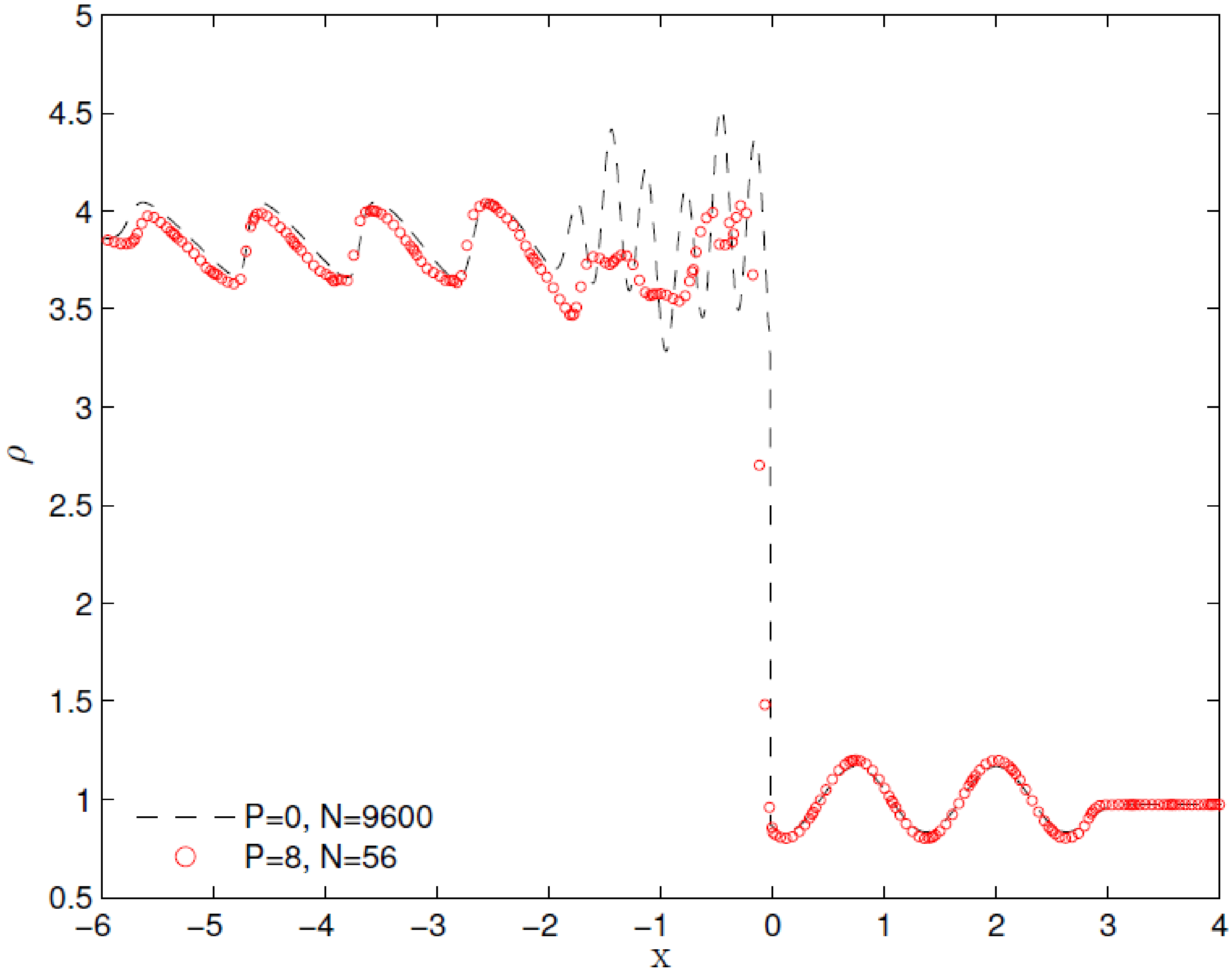
Results

Shu-Osher shock-turbulence interaction; P=8;N=56



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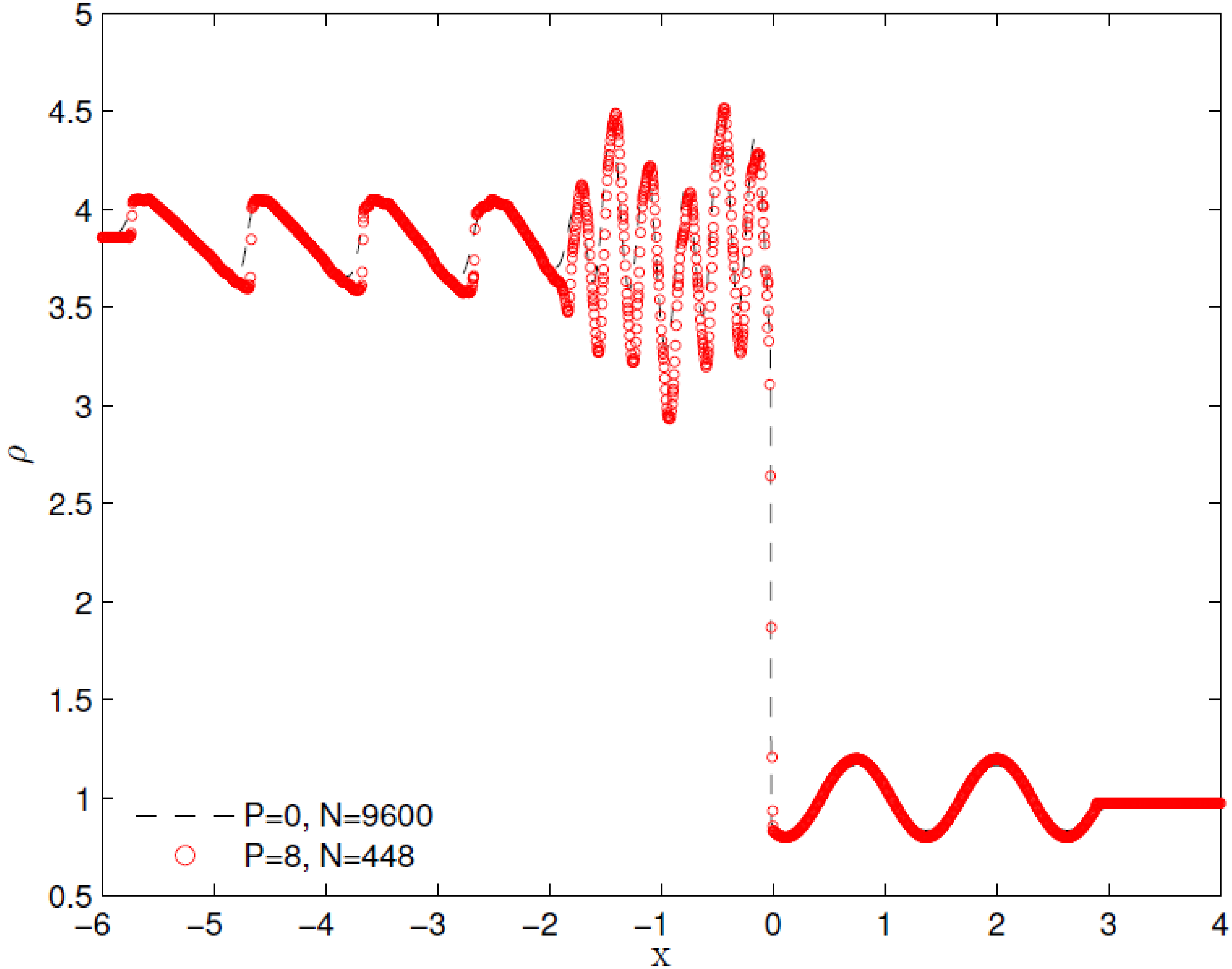
Results

Shu-Osher shock-turbulence interaction; P=8;N=448



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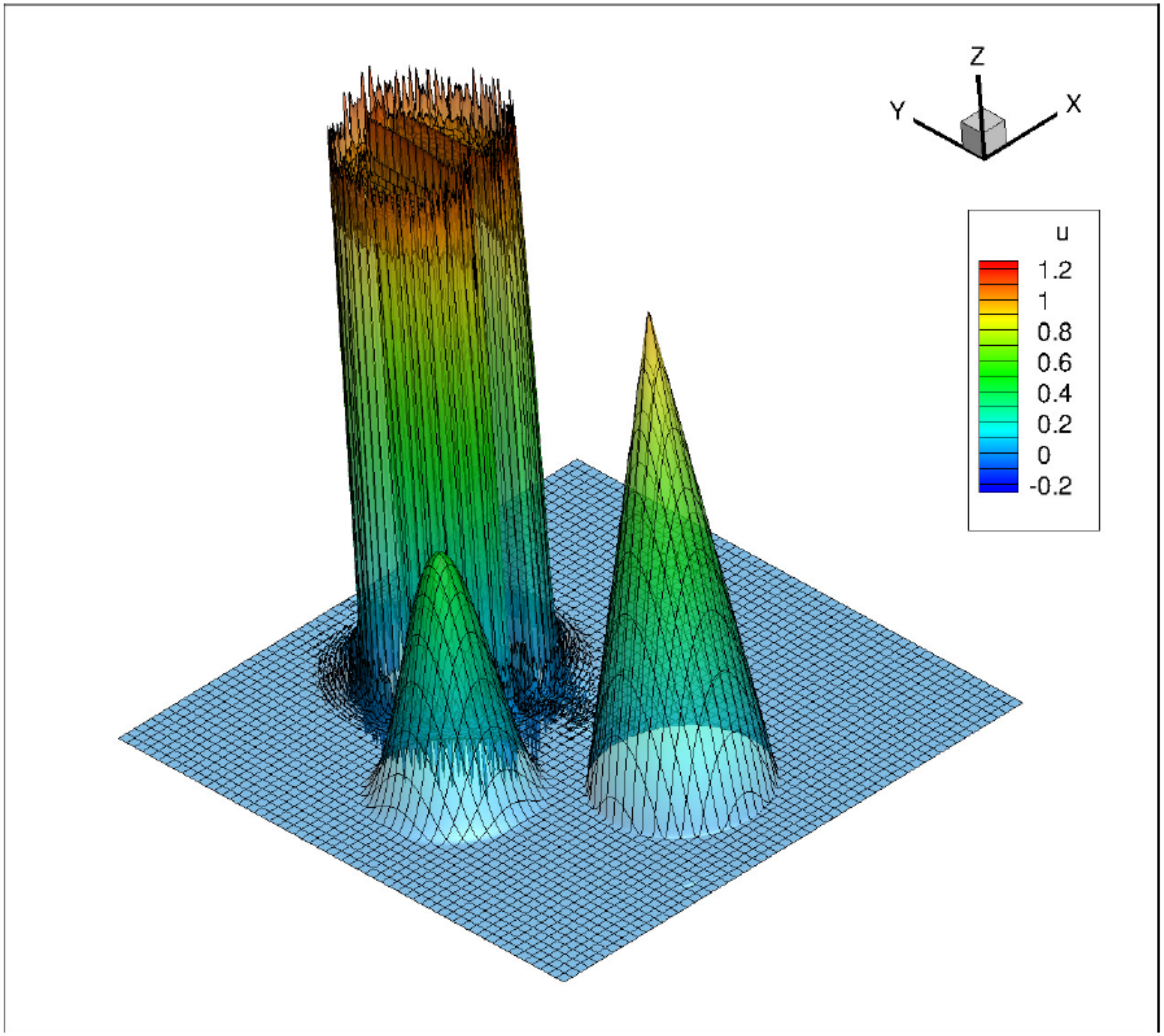
Results

Unfiltered; Rigid body rotation; P=8; N=56x56



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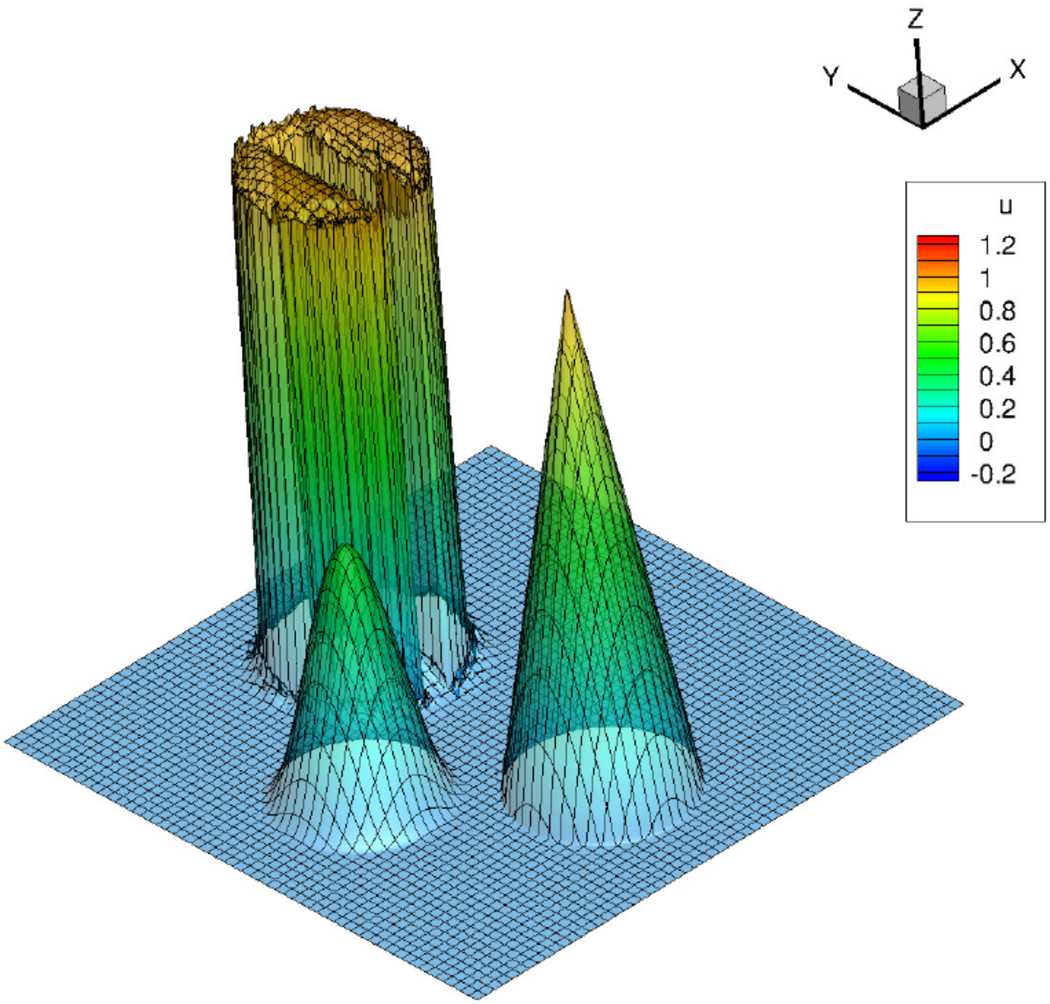
Results

Isotropic box filter; Rigid body rotation; P=8; N=56x56



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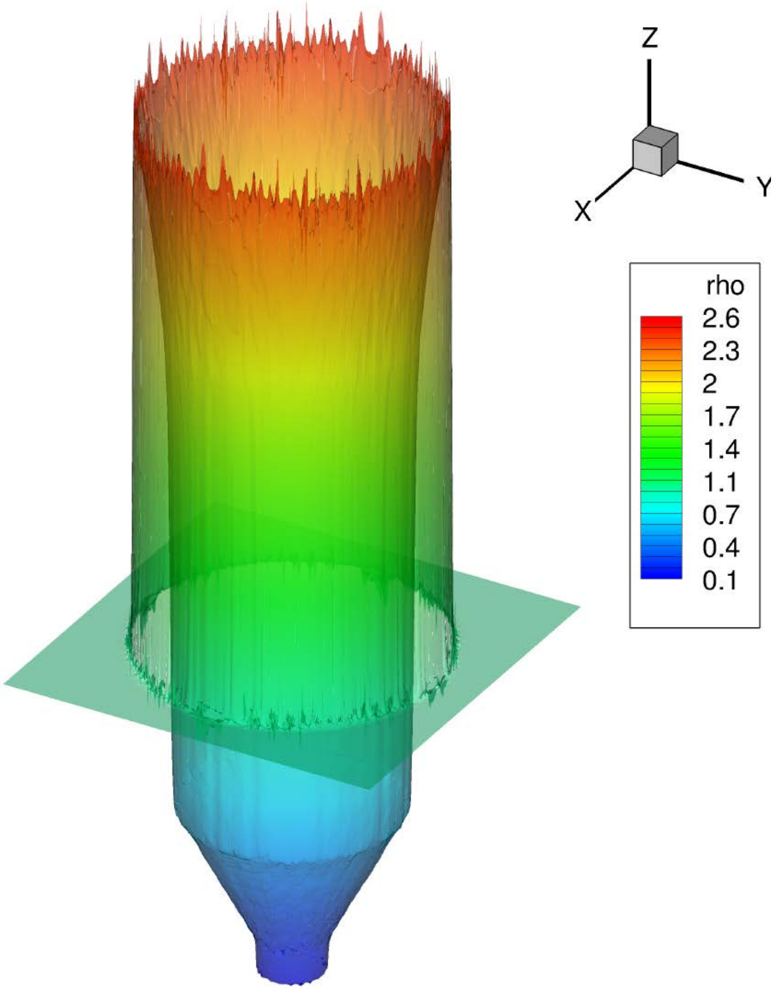
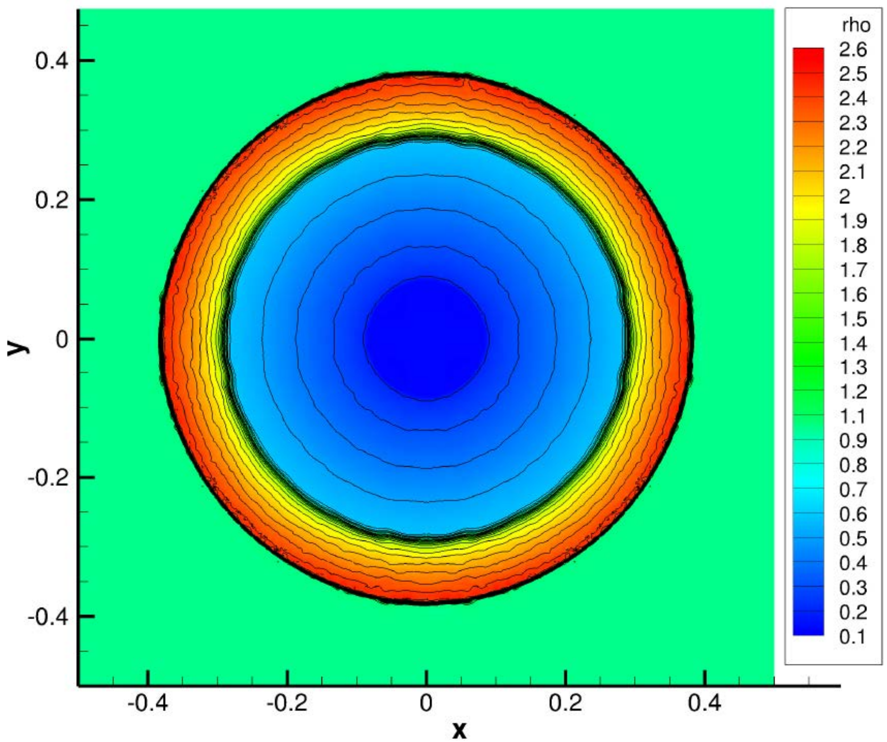
Results

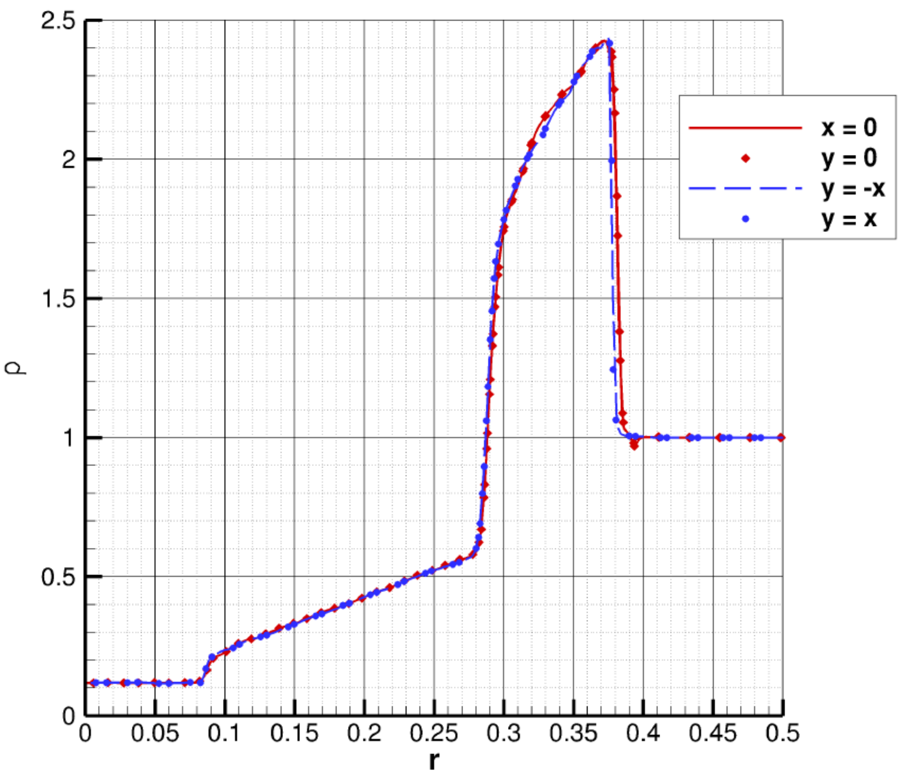
Isotropic box filter; Symmetric Riemann problem;
P=8; N=56x56



Results

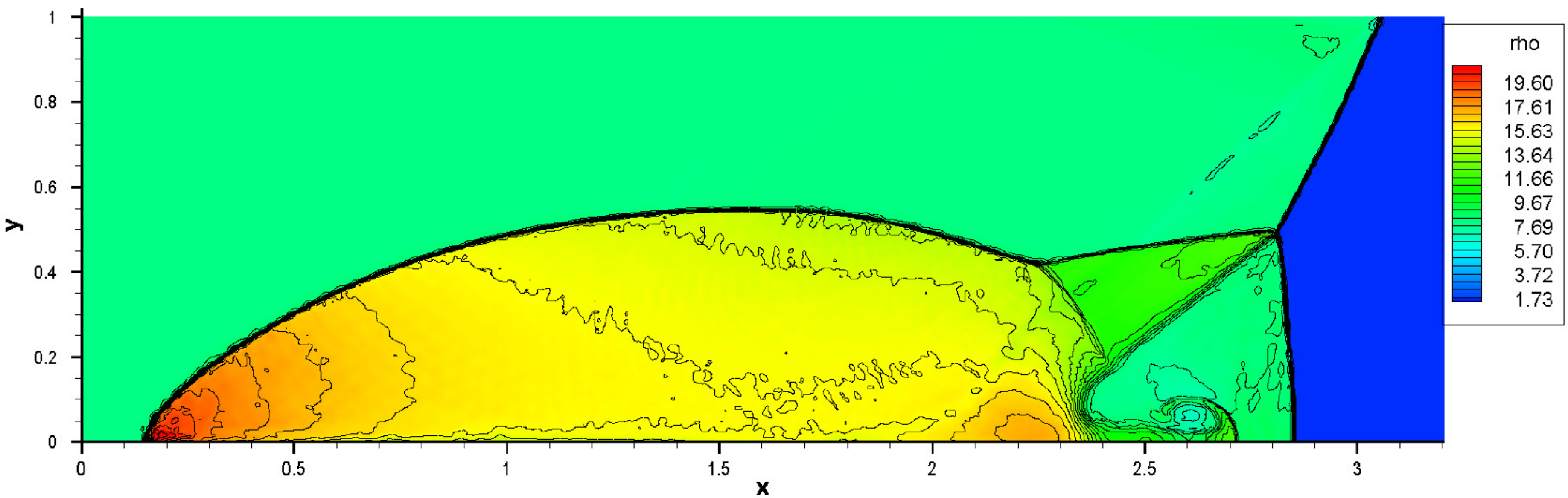
Isotropic box filter; Symmetric Riemann problem; P=8; N=56x56





Results

Isotropic box; double Mach reflection at t = 0.1; P=8; N=56x224; density



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Conclusion

- It is possible to stabilize Flux Reconstruction with artificial dissipation
- Stabilization can be done with local Fourier-spectral filters
- Stabilization integrates seamlessly with matrix-vector multiplication formulation in HiFiLES
- Filters act on high wave-numbers

Future Work

- Implementation in HiFiLES
- Extension to arbitrary polygonal and polyhedral elements
- Parametric study of sensor threshold and filter width parameter

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Questions?